Restrained Total Edge Domination in Graphs

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Abstract

Let $G = (V(G), E(G))$ be a connected graph. A subset $D$ of $E(G)$ is called a restrained total edge dominating set of $G$ if every edge in $E(G)$ is adjacent to an edge in $D$ and every edge not in $D$ is adjacent to another edge not in $D$. The restrained total edge domination number of $G$ denoted by $\gamma_{rte}(G)$, is the minimum cardinality of a restrained total edge dominating set of $G$. Any restrained total edge dominating set of $G$ with cardinality $\gamma_{rte}(G)$ is referred to as a $\gamma_{rte}$-set of $G$. In this paper, we investigate the concept of restrained total edge domination in a graph and obtain some results involving the concepts of edge domination, total edge domination and restrained total edge domination.

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1 Introduction

Let $G = (V(G), E(G))$ be a graph. A graph $G$ has an order $|V(G)| = n$ and size $|E(G)| = m$. The set of vertices in a graph $G$ where no two of which are adjacent is called an independent set. For any vertex $v \in V(G)$, the open neighborhood of $v$, is defined by $N_G(v) = \{u \in V(G) | uv \in E(G)\}$ and the set $N_G[v] = N_G(v) \cup \{v\}$, is the closed neighborhood of $v$. If $S \subseteq V(G)$, then the open neighborhood of $S$ is the set $N_G(S) = \bigcup_{v \in S} N_G(v)$. The closed neighborhood of $S$ is $N_G[S] = N_G(S) \cup S$. A vertex $v \in V(G)$ of degree 1 is called a leaf or end-vertex. The vertex $u \in V(G)$ such that $v \in N_G(u)$, where $v$ is a leaf, is called a support vertex or stem. A subset $S$ of $V(G)$ is a dominating set of $G$ if for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $uv \in E(G)$, that is, $N_G[S] = V(G)$. The domination number of $G$ denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of $G$. Any dominating set of $G$ with cardinality $\gamma(G)$ is referred to as a $\gamma$-set of $G$.

If $e = uv$ is an edge of a graph $G$, then $e$ is incident with vertices $u$ and $v$. In this case, we also say that $u$ and $v$ are incident with $e$. Two edges $e_1$ and $e_2$ which are incident with a common vertex $v$ are said to be adjacent edges. If $D$ is a set of edges in $G$, then the set $V_D \subseteq V(G)$ is given by $V_D = \{v \in V(G) | uv \in D \text{ for some } u \in V(G)\}$. The subgraph $G_D$ of $G$ generated by $D \subseteq E(G)$ is defined by $G_D = (V_D, D)$. A subset $D$ of $E(G)$ is an edge dominating set of $G$ if every edge not in $D$ is adjacent to some edge in $D$. The edge domination number of $G$ denoted by $\gamma_e(G)$, is the minimum cardinality of an edge dominating set of $G$. Any edge dominating set of $G$ with cardinality $\gamma_e(G)$ is referred to as a $\gamma_e$-set of $G$. The concept of edge domination was introduced by Mitchell and Hedetniemi [6] in 1977 and studied in [1, 3, 4, 5, 8, 10].

Arunagam and Velammal [2] initiated the study of total edge domination in graphs in 1997. A subset $D$ of $E(G)$ is a total edge dominating set of $G$ if every edge in $E(G)$ is adjacent to an edge in $D$. The total edge domination number of $G$ denoted by $\gamma_{te}(G)$, is the minimum cardinality of a total edge dominating set of $G$. Any total edge dominating set of $G$ with cardinality $\gamma_{te}(G)$ is referred to as a $\gamma_{te}$-set of $G$. Other studies concerning the concept of total edge domination, with some variations are investigated in [7, 9]. A subset $D$ of $E(G)$ is a restrained edge dominating set of $G$ if every edge not in $D$ is adjacent to an edge in $D$ and to another edge not in $D$. The restrained edge domination number of $G$ denoted by $\gamma_{re}(G)$, is the minimum cardinality of a restrained edge dominating set of $G$. Any restrained edge dominating set of $G$ with cardinality $\gamma_{re}(G)$ is referred to as a $\gamma_{re}$-set of $G$. A subset $D$ of $E(G)$ is called a restrained total edge dominating set of $G$ if every edge in $E(G)$ is adjacent to an edge in $D$ and every edge not in $D$ is adjacent to another edge not in $D$. The restrained total edge domination number of $G$ denoted by
\(\gamma_{rte}(G)\), is the minimum cardinality of a restrained total edge dominating set of \(G\). Moreover, any restrained total edge dominating set of \(G\) with cardinality \(\gamma_{rte}(G)\) is referred to as a \(\gamma_{rte}\)-set of \(G\).

## 2 Restrained Total Edge Domination Numbers of Some Graphs

**Remark 2.1** An edge dominating set \(D\) of \(G\) is a restrained edge dominating set of \(G\) if \(D = E(G)\) or \(G_{E(G)\setminus D}\) has no component isomorphic to \(K_2\).

**Remark 2.2** Every restrained total edge dominating set of a connected graph \(G\) is a total edge dominating set of \(G\).

**Remark 2.3** Let \(G\) be a graph having no component isomorphic to \(K_2\). Then \(G\) has a restrained total edge dominating set \(D\).

**Remark 2.4** Let \(G\) be a connected graph. If \(D\) is a restrained total edge dominating set of \(G\), then \(|D| \geq 2\). Hence, \(\gamma_{rte}(G) \geq 2\).

**Remark 2.5** For a connected graph \(G\) of order \(n \geq 3\), we have

\[2 \leq \gamma_{te}(G) \leq \gamma_{rte}(G) \leq n.\]

**Remark 2.6** Let \(G\) be a graph and let \(D\) be a restrained total edge dominating set of \(G\). If \(v\) is the only leaf connected to a support vertex \(u\), then \(e = uv \in D\).

**Theorem 2.7** Let \(a\) and \(k\) be positive integers such that \(2 \leq a \leq \left\lceil \frac{k}{2} \right\rceil\). Then there exists a connected graph \(G\) such that \(|E(G)| = k\) and \(\gamma_{rte}(G) = a\).

**Proof:** Consider the following cases:

Case 1: \(a\) is even

Let \(a = 2n\), where \(n \geq 1\). Let \(H\) be the union of \(n\) copies of \(P_3\) (\(H\) consists of \(n\) components each isomorphic to \(P_3\)). Set \(m = k - 2a + 2 \geq 2\) and let \(C = \{(x_i, y_i, z_i) | i = 1, 2, ..., n\}\) be a component of \(H\). Let \(G\) be the graph obtained from \(H\) by adding the edges \(z_1w_1, x_2w_1, z_2w_2, x_3w_2, ...,\) and \(z_{n-1}w_{n-1}, x_nw_{n-1}\) and \(z_nv_1, z_nv_2, ...,\) and \(z_nv_m\) (see Figure 1). The set \(S = \{x_1y_1, y_1z_1\} \cup \{x_2y_2, y_2z_2\} \cup ... \cup \{x_{n-1}y_{n-1}, y_{n-1}z_{n-1}\} \cup \{x_ny_n, y_nz_n\} = \cup \{x_iy_i, y_iz_i | i = 1, 2, ..., n\}\) is a \(\gamma_{rte}\)-set of \(G\) by Remark 2.6. Hence, \(\gamma_{rte}(G) = |S| = 2n = a\). Also, \(|E(G)| = 2n + 2(n - 1) + m = 2a - 2 + k - 2a + 2 = k\).
Case 2: $a$ is odd

Let $a = 2n+1$, where $n \geq 1$. Let $H_1$ be the union of $n$ copies of $P_3$ and a copy of $P_4$. Again, let $m = k - 2a + 3 \geq 2$ and let $C = \{(x_i, y_i, z_i, \bar{w}) | i = 1, 2, \ldots, n\}$ be a component of $H_1$. Let $G$ be the graph obtained from $H_1$ by adding the edges $z_1w_1, x_2w_1, z_2w_2, x_3w_2, \ldots$, and $z_{n-1}w_{n-1}, x_nw_{n-1}$ and $\bar{w}v_1, \bar{w}v_2, \ldots$, and $\bar{w}v_m$ (see Figure 2). The set $S = \{x_1y_1, y_1z_1\} \cup \{x_2y_2, y_2z_2\} \cup \ldots \cup \{x_{n-1}y_{n-1}, y_{n-1}z_{n-1}\} \cup \{x_ny_n, y_nz_n, z_n\}$ is a $\gamma_{te}$-set of $G$ by Remark 2.6. Hence, $\gamma_{te}(G) = |S| = 2n + 1 = a$. Also, $|E(G)| = 2(n - 1) + 2(n - 1) + 3 + m = 4n - 1 + k - 2a + 3 = 2(2n + 1) + k - 2a = 2a + k - 2a = k$.

This proves the assertion. $\blacksquare$

**Theorem 2.8** Let $G$ be a graph without isolated vertices. A subset $D$ of $E(G)$ is an edge dominating set of $G$ if and only if $V(G) \setminus V_D$ is an independent set.

**Proof:** Suppose $D$ is an edge dominating set of $G$. Let $x, y \in V(G) \setminus V_D$ such that $x \neq y$. Then $xy \notin E(G)$ (otherwise, $e = xy$ is not dominated by $D$). Thus, $V(G) \setminus V_D$ is an independent set of $G$.

Conversely, suppose that $V(G) \setminus V_D$ is an independent set of $G$. Let $e = uv \in E(G) \setminus D$. Then $u \in V_D$ or $v \in V_D$ (but not both). Assume that $u \in V_D$. Then $v \in V(G) \setminus V_D$ and there exists $w \in V_D$ such that $uw \in D$. Thus, $e = uw$ is adjacent to $uv$. This implies that $D$ is an edge dominating set of $G$. $\blacksquare$

**Theorem 2.9** Let $G$ be a connected graph of order $n \geq 3$. Then $\gamma_{te}(G) = 2$ if and only if there exist distinct vertices $x$, $y$ and $z$ of $G$ such that $\langle\{x, y, z\}\rangle \cong P_3$ or $K_3$ and $V(G) \setminus S$ is an independent set.

**Proof:** Suppose $\gamma_{te}(G) = 2$. Let $D = \{e_1, e_2\}$ be a $\gamma_{te}$-set of $G$. Since $D$ is a total edge dominating set of $G$, $|V_D| = 3$. Let $S = V_D = \{x, y, z\}$. Then $\langle S \rangle \cong P_3$ or $K_3$. Also, $V(G) \setminus S$ is an independent set by Theorem 2.8.

For the converse, suppose that there exist distinct vertices $x$, $y$ and $z$ of $G$ such that $\langle\{x, y, z\}\rangle \cong P_3$ or $K_3$ and $V(G) \setminus S$ is an independent set. Assume
that $e_1 = xy, e_2 = yz \in E(G)$ and let $D = \{xy, yz\}$. Then $D$ is an edge dominating set of $G$ by Theorem 2.8. Moreover, since $e_1$ and $e_2$ are adjacent edges, $D$ is a total edge dominating set of $G$. Therefore, $\gamma_{te}(G) = |D| = 2$. □

**Theorem 2.10** Let $G$ be a connected graph of order $n \geq 3$. Then $\gamma_{rte}(G) = 2$ if and only if $G \neq K_3$ and there exists $S \subseteq V(G)$ with $|S| = 3$ satisfying the following conditions:

(i) $\langle S \rangle \cong P_3$ or $K_3$;

(ii) $V(G) \setminus S$ is an independent set of $G$;

(iii) If $\langle S \rangle \cong P_3$ and if $w \in S$ is a support vertex to some vertex $u \in V(G) \setminus S$, then $|N_G(w) \cap (V(G) \setminus S)| \geq 2$; and

(iv) If $\langle S \rangle \cong K_3$ and every element of $S$ is a support vertex, then there exists $w \in S$ such that $|N_G(w) \cap (V(G) \setminus S)| \geq 2$.

**Proof:** Suppose $\gamma_{rte}(G) = 2$. Then $G \neq K_3$ and $\gamma_{te}(G) = 2$ by Remark 2.5. Hence, by Theorem 2.9, conditions (i) and (ii) hold. Let $D = \{e_1, e_2\}$ be a restrained total edge dominating set of $G$ with $V_D = S$. Suppose $\langle S \rangle \cong P_3$ and $w \in S$ is a support vertex to some $u \in V(G) \setminus S$. Then $uw \notin D$. Since $D$ is a restrained total edge dominating set of $G$, there exists $ab \notin D$ which is adjacent to $uw$. This implies that either $u$ or $w$ is incident to $ab$. Since $u$ is a leaf, $w$ must be incident to $ab$. Assume that $a = w$. Then $b \in V(G) \setminus S$. Hence, $u, b \in V(G) \setminus S$, showing that $|N_G(w) \cap (V(G) \setminus S)| \geq 2$. Thus, (iii) holds. Suppose $\langle S \rangle \cong K_3$. Suppose that every element of $S$ is a support vertex. Suppose further that $|N_G(w) \cap (V(G) \setminus S)| = 1$, for all $w \in S$. Assume that $q \in S$ is the vertex incident to both $e_1$ and $e_2$. Let $p \in V(G) \setminus S$ such that $pq \in E(G)$. Then $\{e_1, e_2\}$ cannot be a restrained total edge dominating set of $G$ because $pq$ is not adjacent to any edge in $\langle E(G) \setminus D \rangle$, contrary to our assumption. Thus, there exists $w \in S$ such that $|N_G(w) \cap (V(G) \setminus S)| \geq 2$, showing that (iv) holds.

For the converse, suppose that $G \neq K_3$ and there exists $S \subseteq V(G)$ with $|S| = 3$ satisfying (i), (ii), (iii) and (iv). Suppose first that $\langle S \rangle \cong P_3$, say $\langle S \rangle = \{x, y, z\}$. Let $D = \{xy, yz\}$. Since $V(G) \setminus V_D = V(G) \setminus S$ is an independent set, $D$ is a total edge dominating set of $G$ by Theorem 2.8. Let $ab \in E(G) \setminus D$. 

![Figure 2](image-url)
Then \( a \in V(G) \setminus S \) or \( b \in V(G) \setminus S \) (but not both). Assume that \( a \in V(G) \setminus S \). Then \( b \in S \). If \( a \) is not a leaf, then \( ac \in E(G) \) for some \( c \in S \setminus \{b\} \). Clearly, \( ac \in E(G) \setminus D \) and \( ac \) is adjacent to \( ab \). Suppose \( a \) is a leaf. Then \( b \in S \) is a support vertex. Hence, by (iii), there exists \( d \in (V(G) \setminus S) \cap N_G(b) \). Thus, \( bd \in E(G) \setminus D \) and is adjacent to \( ab \). This shows that \( D \) is a restrained total edge dominating set of \( G \).

Finally, suppose that \( \langle S \rangle \cong K_3 \), where \( S = \{x, y, z\} \). Suppose that one of \( x, y \) and \( z \), say \( x \), is not a support vertex. Let \( D = \{xy, xz\} \). Then \( D \) is a total edge dominating set of \( G \). Let \( e = uv \in E(G) \setminus D \). Assume that \( u \in V(G) \setminus S \) and \( v \in S = V_D \). Suppose first that \( v = x \). Since \( x \) is not a support vertex (that is, \( u \) is not a leaf), there exists \( q \in (S \setminus \{x\}) \cap N_G(u) \). Thus, \( uv \in E(G) \setminus D \) and is adjacent to \( uv \). Suppose \( v \neq x \). Then \( uv \) is adjacent to \( yz \in E(G) \setminus D \). Hence, \( D \) is a restrained total edge dominating set of \( G \). Suppose now that \( x, y \) and \( z \) are support vertices. Then by (iv), one of them, say \( x \), satisfies \( |N_G(x) \cap (V(G) \setminus S)| \geq 2 \). Let \( D_1 = \{xy, yz\} \). Then \( D_1 \) is a restrained total edge dominating set of \( G \). Accordingly, \( \gamma_{rte}(G) = 2 \).

Clearly, \( \gamma_{rte}(K_{1,3}) = 3 \). Also, by Theorem 2.10, \( \gamma_{rte}(K_{1,n-1}) = 2 \) for \( n \in \{3, 5, 6, \ldots\} \). The next result summarizes these facts.

**Corollary 2.11** Let \( n \geq 3 \). Then

\[
\gamma_{rte}(K_{1,n-1}) = \begin{cases} 
3, & n = 4 \\
2, & n \neq 4 
\end{cases}
\]

**Lemma 2.12** Let \( n \) be an integer with \( n \geq 2 \). Then \( D \subseteq E(G) \) is a restrained edge dominating set of \( K_n \) if and only if \( |V(K_n) \setminus V_D| \leq 1 \) and \( D = E(G) \) or \( G_{E(G),D} \) has no component isomorphic to \( K_2 \).

**Proof:** Suppose that \( D \) is a restrained edge dominating set of \( K_n \). Suppose further that \( |V(K_n) \setminus V_D| \geq 2 \), say \( x, y \in V(K_n) \setminus V_D \). Then \( xy \) is not dominated by \( D \), contrary to the assumption that \( D \) is a restrained edge dominating set of \( K_n \). Thus, \( |V(K_n) \setminus V_D| \leq 1 \). Also, \( D = E(G) \) or \( G_{E(G),D} \) has no component isomorphic to \( K_2 \) by Remark 2.1.

Conversely, suppose that \( |V(K_n) \setminus V_D| \leq 1 \) and \( D = E(G) \) or \( G_{E(G),D} \) has no component isomorphic to \( K_2 \). Let \( e = xy \in E(G) \setminus D \). Then \( G \neq K_2 \); hence \( n \geq 3 \). If one of \( x \) and \( y \), say \( x \), is not in \( V_D \), then \( V(G) \setminus \{x\} \subseteq V_D \). Hence, there exists \( z \in V(G) \setminus \{x, y\} \) such that \( e' = yz \in D \). Clearly, \( e \) and \( e' \) are adjacent. If \( x, y \in V_D \), then \( e \) is dominated by \( D \). Hence, \( D \) is a restrained edge dominating set of \( G = K_n \).

**Theorem 2.13** Let \( n \) be a positive integer with \( n \geq 3 \). Then

\[
\gamma_{rte}(K_n) = \begin{cases} 
3, & n = 3 \\
 n - \left\lceil \frac{n}{3} \right\rceil, & n \geq 4 
\end{cases}
\]
Proof: If \( n = 3 \), then \( \gamma_{rte}(K_3) = 3 \). Suppose that \( n \geq 4 \) and let \( V(K_n) = \{v_1, v_2, \ldots, v_n\} \). Consider the edges \( e_1 = v_1v_2 \) and \( e_2 = v_2v_3 \). Then the edges of \( K_n \) not dominated by \( e_1 \) and \( e_2 \) are exactly the edges of \( \{v_4, v_5, \ldots, v_{n-1}, v_n\} \) \( \equiv K_{n-3} \). Consider the edges \( e_3 = v_4v_5 \) and \( e_4 = v_5v_6 \). Then the edges of \( K_n \) not dominated by \( e_1, e_2, e_3 \) and \( e_4 \) are the edges of \( \{v_7, v_8, \ldots, v_{n-1}, v_n\} \) \( \equiv K_{n-6} \). Continuing in this manner, we obtain a set of edges \( e_1, e_2, \ldots, e_k \) that either dominates the edges of \( K_n \) or does not dominate the single edge \( v_{n-1}v_n \). By our choice of the edges (minimizing the number of edges not dominated in each stage), it follows that \( D = \{e_1, e_2, \ldots, e_k\} \) or \( D' = \{e_1, e_2, \ldots, e_k, v_{n-1}v_n\} \) is a \( \gamma_{rte} \)-set of \( G = K_n \). Consider the following cases:

Case 1: \( D = \{e_1, e_2, \ldots, e_k\} \) is a \( \gamma_{rte} \)-set of \( K_n \)
By Lemma 2.12, \( |V(K_n)\setminus V_D| \leq 1 \). Suppose that \( |V(K_n)\setminus V_D| = 0 \), that is, \( V_D = V(K_n) \). Then \( n \equiv 0 \) mod \( (3) \), that is, \( n = 3t \) for some positive integer \( t \). It follows that \( |D| = k = 2t = 3t - t = n - \left\lceil \frac{n}{3} \right\rceil \). Thus, \( \gamma_{rte}(K_n) = n - \left\lceil \frac{n}{3} \right\rceil \).

Suppose that \( |V(K_n)\setminus V_D| = 1 \). Then \( n \equiv 1 \) mod \( (3) \), that is, \( n = 3t + 1 \) for some non-negative integer \( t \). Thus, \( |D| = k = 2t = n - \left\lceil \frac{n}{3} \right\rceil \). Hence, \( \gamma_{rte}(K_n) = n - \left\lceil \frac{n}{3} \right\rceil \).

Case 2: \( D' = \{e_1, e_2, \ldots, e_k, v_{n-1}v_n\} \) is a \( \gamma_{rte} \)-set of \( K_n \)
Then \( D = \{e_1, e_2, \ldots, e_k\} \) does not dominate the single edge \( v_{n-1}v_n \). This implies that \( n \equiv 2 \) mod \( (3) \), that is, \( n = 3t + 2 \) for some positive integer \( t \). It follows that \( |D'| = k + 1 = 2t + 1 = n - \left\lceil \frac{n}{3} \right\rceil \). Thus, \( \gamma_{rte}(K_n) = n - \left\lceil \frac{n}{3} \right\rceil \). \hspace{1cm} \( \blacksquare \)

Theorem 2.14 For any path \( P_n \) with \( n \geq 3 \) vertices,

\[
\gamma_{rte}(P_n) = (n - 1) - 2 \left\lfloor \frac{n - 3}{4} \right\rfloor.
\]

Proof: Let \( D \) be a \( \gamma_{rte} \)-set of \( P_n \) and let \( E(P_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\} \). Note that \( v_1v_2, v_2v_3, v_{n-2}v_{n-1}, v_{n-1}v_n \in D \) and any non-trivial component of \( G_{E(P_n)\setminus D} \) is of size exactly two. Suppose there are \( k \) non-trivial components in \( G_{E(P_n)\setminus D} \). Then \( 2k + 2(k + 1) \leq n - 1 \), that is, \( k \leq \frac{n - 3}{4} \). Thus, \( |D| = (n - 1) - 2k \geq (n - 1) - 2 \left\lfloor \frac{n - 3}{4} \right\rfloor \). Hence, \( \gamma_{rte}(P_n) \geq (n - 1) - 2 \left\lfloor \frac{n - 3}{4} \right\rfloor \).

Next, consider the following cases:

Case 1: \( n - 3 \equiv 0 \) mod \( (4) \)
Then \( n - 3 = 4t \) for some non-negative integer \( t \). Now, the set \( \{v_{4i+1}v_{4i+2}, v_{4i+2}v_{4i+3} | i = 0, 1, 2, \ldots, t - 1\} \cup \{v_{4t+1}v_{4t+2}, v_{4t+2}v_{4t+3}\} \) is a restrained total edge dominating set of \( P_n \) with cardinality \( (n - 1) - 2 \left\lfloor \frac{n - 3}{4} \right\rfloor = 2t + 1 \). Hence, \( \gamma_{rte}(P_n) \leq (n - 1) - 2 \left\lfloor \frac{n - 3}{4} \right\rfloor \), showing that \( \gamma_{rte}(P_n) = (n - 1) - 2 \left\lfloor \frac{n - 3}{4} \right\rfloor \).

Case 2: \( n - 3 \equiv 1 \) mod \( (4) \)
Then \( n - 3 = 4t + 1 \) for some positive integer \( t \) and \( (n - 1) - 2 \left\lfloor \frac{n - 3}{4} \right\rfloor = 2t + 3 \). The set \( \{v_{4i+1}v_{4i+2}, v_{4i+2}v_{4i+3} | i = 0, 1, 2, \ldots, t - 1\} \cup \{v_{4t+1}v_{4t+2}, v_{4t+2}v_{4t+3}, v_{4t+3}v_{4t+4}\} \)
is a restrained total edge dominating set of $P_n$ with cardinality $2t + 3$. Thus, 
\[ \gamma_{rte}(P_n) \leq (n - 1) - 2 \left\lfloor \frac{n - 3}{4} \right\rfloor. \] Therefore, 
\[ \gamma_{rte}(P_n) = (n - 1) - 2 \left\lfloor \frac{n - 3}{4} \right\rfloor. \]

Case 3: $n - 3 \equiv 2 \text{mod}(4)$

Then $n - 3 = 4t + 2$ for some positive integer $t$ and $(n - 1) - 2 \left\lfloor \frac{n - 3}{4} \right\rfloor = 2(t + 2)$. The set
\[ \{v_{4i+1}v_{4i+2}, v_{4i+2}v_{4i+3} \mid i = 0, 1, 2, ..., t - 1\} \]
\[ \cup \{v_{4t+1}v_{4t+2}, v_{4t+2}v_{4t+3}, v_{4t+3}v_{4t+4}, v_{4t+4}v_{4t+5}\} \]
is a restrained total edge dominating set of $P_n$ with cardinality $2(t + 2)$ and so 
\[ \gamma_{rte}(P_n) \leq (n - 1) - 2 \left\lfloor \frac{n - 3}{4} \right\rfloor. \] Hence, 
\[ \gamma_{rte}(P_n) = (n - 1) - 2 \left\lfloor \frac{n - 3}{4} \right\rfloor. \]

Case 4: $n - 3 \equiv 3 \text{mod}(4)$

Then $n - 3 = 4t + 3$ for some positive integer $t$ and $(n - 1) - 2 \left\lfloor \frac{n - 3}{4} \right\rfloor = 2t + 5$. Since 
\[ \{v_{4i+1}v_{4i+2}, v_{4i+2}v_{4i+3} \mid i = 0, 1, 2, ..., t - 1\} \]
\[ \cup \{v_{4t+1}v_{4t+2}, v_{4t+2}v_{4t+3}, v_{4t+3}v_{4t+4}, v_{4t+4}v_{4t+5}, v_{4t+5}v_{4t+6}\} \]
is a restrained total edge dominating set of $P_n$ with cardinality $2t + 5$, 
\[ \gamma_{rte}(P_n) \leq (n - 1) - 2 \left\lfloor \frac{n - 3}{4} \right\rfloor. \] Hence, 
\[ \gamma_{rte}(P_n) = (n - 1) - 2 \left\lfloor \frac{n - 3}{4} \right\rfloor. \]

\[ \Box \]

**Theorem 2.15** For any cycle $C_n$ with $n \geq 3$ vertices,
\[ \gamma_{rte}(C_n) = n - 2 \left\lfloor \frac{n}{4} \right\rfloor. \]

**Proof:** Let $D$ be a $\gamma_{rte}$-set of $C_n$ and let $E(C_n) = \{v_1v_2, v_2v_3, ..., v_{n-1}v_n, v_nv_1\}$. Note that $v_1v_2, v_2v_3 \in D$ and any non-trivial component of $G_{E(C_n) \setminus D}$ is of size exactly two. Suppose there are $k$ non-trivial components in $G_{E(C_n) \setminus D}$. Then $2(2k) \leq n$, that is, $k \leq \frac{n}{4}$. Thus, $|D| = n - 2k \geq n - 2 \left\lfloor \frac{n}{4} \right\rfloor$. Hence, 
\[ \gamma_{rte}(C_n) \geq n - 2 \left\lfloor \frac{n}{4} \right\rfloor. \]

Next, consider the following cases:

Case 1: $n \equiv 0 \text{mod}(4)$

Then $n = 4t$ for some positive integer $t$ and $n - 2 \left\lfloor \frac{n}{4} \right\rfloor = n - 2t = 2t$. Clearly, the set 
\[ \{v_{4i+1}v_{4i+2}, v_{4i+2}v_{4i+3} \mid i = 0, 1, 2, ..., t\} \]
is a restrained total edge dominating set of $C_n$ with cardinality $2t = n - 2 \left\lfloor \frac{n}{4} \right\rfloor$. Consequently, 
\[ \gamma_{rte}(C_n) \leq n - 2 \left\lfloor \frac{n}{4} \right\rfloor. \] Hence, 
\[ \gamma_{rte}(C_n) = n - 2 \left\lfloor \frac{n}{4} \right\rfloor. \]

Case 2: $n \equiv 1 \text{mod}(4)$

Then $n = 4t + 1$ for some positive integer $t$ and $n - 2 \left\lfloor \frac{n}{4} \right\rfloor = n - 2t = 2t + 1$. Since the set 
\[ \{v_{4i+1}v_{4i+2}, v_{4i+2}v_{4i+3} \mid i = 0, 1, 2, ..., t - 1\} \cup \{v_{4t+1}v_1\} \]
is a restrained total edge dominating set of $C_n$ with cardinality $2t + 1 = n - 2 \left\lfloor \frac{n}{4} \right\rfloor$, 
\[ \gamma_{rte}(C_n) \leq n - 2 \left\lfloor \frac{n}{4} \right\rfloor. \] Hence, 
\[ \gamma_{rte}(C_n) = n - 2 \left\lfloor \frac{n}{4} \right\rfloor. \]
Case 3: $n \equiv 2 \pmod{4}$
Then $n = 4t+2$ for some positive integer $t$ and $n-2 \left\lfloor \frac{n}{4} \right\rfloor = n-2t = 2(t+1)$. Since \( \{v_{4i+1}v_{4i+2}, v_{4i+2}v_{4i+3} | i = 0, 1, 2, \ldots, t-1\} \cup \{v_{4t+1}v_{4t+2}, v_{4t+2}v_1\} \) is a restrained total edge dominating set of $C_n$ with cardinality $2(t+1) = n-2 \left\lfloor \frac{n}{4} \right\rfloor$, \( \gamma_{rte}(C_n) \leq n-2 \left\lfloor \frac{n}{4} \right\rfloor \). Hence, \( \gamma_{rte}(C_n) = n-2 \left\lfloor \frac{n}{4} \right\rfloor \).

Case 4: $n \equiv 3 \pmod{4}$
Then $n = 4t+3$ for some non-negative integer $t$ and $n-2 \left\lfloor \frac{n}{4} \right\rfloor = n-2t = 2t+3$. Since \( \{v_{4i+1}v_{4i+2}, v_{4i+2}v_{4i+3} | i = 0, 1, 2, \ldots, t-1\} \cup \{v_{4t+1}v_{4t+2}, v_{4t+2}v_{4t+3}, v_{4t+3}v_1\} \) is a restrained total edge dominating set of $C_n$ with cardinality $2t+3 = n-2 \left\lfloor \frac{n}{4} \right\rfloor$, \( \gamma_{rte}(C_n) \leq n-2 \left\lfloor \frac{n}{4} \right\rfloor \). Hence, \( \gamma_{rte}(C_n) = n-2 \left\lfloor \frac{n}{4} \right\rfloor \). ■

References


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