Study of Renewable Systems Based on Functional Operators with Shift

Oleksandr Karelin, Manuel Gonzalez-Hernandez, Norberto Hernandez-Romero and Juan Seck-Tuoh-Mora

Hidalgo State University, Institute of Basic Sciences and Engineering
Carretera Pachuca-Tulancingo, Km.4.5, C.P.42184, Pachuca Hgo, Mexico

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Abstract

In previous works we proposed a method for the study of systems with one renewable resource. The separation of the individual and the group parameters and the discretization of time led us to scalar linear functional equations with shift. Cyclic models, in which the initial state of the system coincides with the final state, were considered. In this work, we present cyclic models for systems with two renewable resources. In modeling, the interactions and the reciprocal influences between these two resources are taken into account. We applied our results on invertibility of the functional operators with shift to the study of the balance equations. The equilibrium state of the system is found.

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1 Introduction

The interest towards the study of functional operators with shift was stipulated by the development of solvability theory and Fredholm theory of characteristic singular integral equation with non-Carleman shift [3, 4, 5]. Our motivation
to return to these investigations has its reasons. On modeling systems with renewable resources, linear functional operators with shift appear \[6, 2\]. These operators are the adequate mathematical instrument for the investigation of such systems.

In Section 2, we construct a cyclic model of a system with two renewable resources, in which the initial state of the system coincides with the final state. Terms that take into account the reciprocal influence of these resources on each other are introduced. The system of balance equations of the cyclic model is represented through linear functional equations with shift.

In Section 3, using some known results \[7\] on the invertibility of the functional operator with shift in weighted Hölder spaces, a method for solution of the system of balance equations is offered. The equilibrium state of the system is found.

2 Cyclic model of a system with two renewable resources

Let \( S \) be a system with two resources \( \lambda_1, \lambda_2 \), and let \( T \) be a time interval. Let \( t_0 \) be the initial time and \( S \) the system under consideration.

The initial state of the system \( S \) at time \( t_0 \) is represented by density functions of the distribution of the group parameter by the individual parameter for each resource \( v(x, t_0) = v(x), 0 < x < x_{max}, \ w(y, t_0) = w(y), 0 < y < y_{max} \).

We will now analyze the system’s evolution. In the course of time, the elements of the system can change their individual parameter - e.g. fish can change their weight and length. Modifications in the distribution of the group parameters by the individual parameters is represented by a displacement. The state of the system \( S \) at the time \( t = t_0 + T \) is:

\[
v(x, t_0 + T) = \frac{d}{dx} \alpha(x) \cdot v(\alpha(x)), \quad w(y, t_0 + T) = \frac{d}{dy} \beta(y) \cdot w(\beta(y)). \tag{1}
\]

In the article \[6\], the appearance of derivatives in (1) was explained.

Over the period \( j_0 = [t_0, t_0 + T] \), extractions might be taken from the system as a result of human economic activity; these are represented by summands \( \rho(x), \delta(y) \). If an artificial entrance of elements into the system has taken place, it shall be accounted for by adding terms \( \zeta(x), \xi(y) \). We take natural mortality into account with the coefficients \( c(x), d(y) \).

The process of reproduction will be represented by terms

\[
\sum_{i=1}^{n} P_i p_i(x), \quad P_i = \int_{\nu_{i-1}}^{\nu_i} v(x) dx, \ 0 = \nu_0 < \nu_1 < \ldots < \nu_n = x_{max},
\]
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\[ \sum_{i=1}^{m} Q_i q_i(y), \quad Q_i = \int_{\mu_{i-1}}^{\mu_i} w(y)dy, \quad 0 = \mu_0 < \mu_1 < \ldots < \mu_m = y_{max}. \]

We obtain

\[ v(x, t_0 + T) = c(x) \frac{d}{dx} \alpha(x)v(\alpha(x)) + \rho(x) + \zeta(x) + \sum_{i=1}^{n} P_i p_i(x), \]

\[ w(y, t_0 + T) = d(y) \frac{d}{dy} \beta(y)w(\beta(y)) + \delta(y) + \xi(y) + \sum_{i=1}^{m} Q_i q_i(y). \]

We will account for reciprocal influence of the resources \( \lambda_1 \) and \( \lambda_2 \) by

\[ \sum_{i=1}^{k} R_i r_i(x), \quad R_i = \int_{\gamma_{i-1}}^{\gamma_i} w(y)dy, \quad 0 = \gamma_0 < \gamma_1 < \ldots < \gamma_k = y_{max}, \]

\[ \sum_{i=1}^{l} F_i f_i(y), \quad F_i = \int_{\epsilon_{i-1}}^{\epsilon_i} v(x)dx, \quad 0 = \epsilon_0 < \epsilon_1 < \ldots < \epsilon_l = x_{max}. \]

Thereby, the final state of the system at the moment \([t_0 + T]\) is described as follows:

\[ v(x, t_0 + T) = c(x) \frac{d}{dx} \alpha(x)v(\alpha(x)) + \rho(x) + \zeta(x) + \sum_{i=1}^{n} P_i p_i(x) + \sum_{i=1}^{k} R_i r_i(x), \quad (2) \]

\[ w(y, t_0 + T) = d(y) \frac{d}{dy} \beta(y)w(\beta(y)) + \delta(y) + \xi(y) + \sum_{i=1}^{m} Q_i q_i(y) + \sum_{i=1}^{l} F_i f_i(y). \quad (3) \]

Let our goal be to find the equilibrium state of system \( S \), that is, to find such an initial distribution of the group parameters by the individual parameter \( v(x, t_0), w(x, t_0) \), that after all transformations during the time interval \((t_0, t_0 + T)\), it would coincide with the final distribution:

\[ v(x, t_0) = v(x, t_0 + T) = v(x), \quad w(y, t_0) = w(y, t_0 + T) = w(y). \quad (4) \]

From here, substituting relations (2) and (3) into (4), it follows that

\[ v(x) = c(x) \frac{d}{dx} \alpha(x)v(\alpha(x)) + \rho(x) + \zeta(x) + \sum_{i=1}^{n} P_i p_i(x) + \sum_{i=1}^{k} R_i r_i(x), \quad (5) \]

\[ w(y) = d(y) \frac{d}{dy} \beta(y)w(\beta(y)) + \delta(y) + \xi(y) + \sum_{i=1}^{m} Q_i q_i(y) + \sum_{i=1}^{l} F_i f_i(y). \quad (6) \]

Equations (5), (6) are called equilibrium proportions or balance equations.

A model is called cyclic if the state of system \( S \) at the initial time \( t_0 \) coincides with the state of system \( S \) at the final time \( t_0 + T \).

Without loss of generality, we assume below that \( x_{max} = 1, y_{max} = 1 \).
3 About the space in which the cyclic model is considered and conditions of invertibility of functional operators with shift

A function \( \varphi(x) \) that satisfies the condition on \( J = [0, 1] \),
\[
| \varphi(x_1) - \varphi(x_2) | \leq C \cdot |x_1 - x_2|^{\varsigma}, \quad x_1, x_2 \in J, \varsigma \in (0, 1),
\]
is called a Hölder function with exponent \( \varsigma \) and constant \( C \) on \( J \).

Let \( \varrho \) be a power function which has zeros at the endpoints \( x = 0, 1 \), after being multiplied by \( \varrho(x) \), form a Banach space. Functions of this space are called Hölder functions with weight \( \varrho \): \( H^\varsigma(J, \varrho), J = [0, 1] \). The norm in the space \( H^\varsigma(J, \varrho) \) is defined by
\[
\| f(x) \|_{H^\varsigma(J, \varrho)} = \| \varrho(x)f(x) \|_{H(\varrho)}, \quad \| \varrho(x)f(x) \|_{H(\varrho)} = \| \varrho(x)f(x) \|_{C} + \| \varrho(x)f(x) \|_{\varsigma},
\]
where
\[
\| \varrho(x)f(x) \|_{C} = \max_{x \in J} |\varrho(x)f(x)|,
\]
\[
\| \varrho(x)f(x) \|_{\varsigma} = \sup_{x_1, x_2 \in J, x_1 \neq x_2} \frac{|\varrho(x_1)f(x_1) - \varrho(x_2)f(x_2)|}{|x_1 - x_2|^{\varsigma}}.
\]

Let \( \beta(x) \) be a bijective orientation-preserving displacement on \( J \) if \( x_1 < x_2 \) then \( \beta(x_1) < \beta(x_2) \) for any \( x_1, x_2 \in J \) and let \( \beta(x) \) have only two fixed points: \( \beta(0) = 0, \beta(1) = 1, \beta(x) \neq x, \) when \( x \neq 0, x \neq 1 \). In addition, let \( \beta(x) \) be a differentiable function and \( \frac{d}{dx}\beta(x) \neq 0, x \in J \).

We consider the equation
\[
(A\nu)(x) = f(x), \quad (A\nu)(x) \equiv a(x)(I\nu)(x) - b(x)(\Gamma_{\beta}\nu)(x), \quad x \in [0, 1] \quad (7)
\]
where \( I \) is the identity operator and \( \Gamma_{\beta} \) is the shift: \( (\Gamma_{\beta}\nu)(x) = \nu[\beta(x)] \).

Let functions \( a(x), b(x) \) from operator \( A \) belong to \( H(\nu) \).

We will now formulate conditions of invertibility for operator \( A \) from (7) in the space of Hölder class functions with weight [6].

**Theorem 3.1** Operator \( A \), acting in Banach space \( H^\varsigma(J, \varrho) \), is invertible if the following condition is fulfilled: \( \theta_{\beta}[a(x), b(x), H^\varsigma(J, \varrho)] \neq 0, \quad x \in J \), where the function \( \sigma_{\beta} \) is defined by
\[
\theta_{\beta}[a(x), b(x), H^\varsigma(J, \varrho)] = \begin{cases} 
  a(x), & \text{when } a(0) > [\beta'(0)]^{-\varsigma_0 + \varsigma} |b(0)|; \text{ and, } |a(l)| > [\beta'(l)]^{-\varsigma_1 + \varsigma} |b(l)|; \\
  b(x), & \text{when } a(0) < [\beta'(0)]^{-\varsigma_0 + \varsigma} |b(0)|; \text{ and, } |a(l)| < [\beta'(l)]^{-\varsigma_1 + \varsigma} |b(l)|; \\
  0 & \text{in other cases.}
\end{cases}
\]
4 Analysis of the solvability of the balance equations and finding the equilibrium state of the system

Let $S$ be the system with two resources considered in Section 2. We find the equilibrium state of the system in which the initial distribution of the group parameters by the individual parameters $v(x), w(y), x \in (0,1)$ coincide with the final distribution, after all transformations during the time interval $T$.

We rewrite the balance equations of the cyclic model (5), (6) for system $S$ in the form

$$(Vv)(x) = \sum_{i=1}^{n} P_i p_i(x) + \sum_{i=1}^{k} R_i r_i(x) + g(x), \quad (Vv)(x) = v(x) - c_\alpha(x)v(\alpha(x)) \quad (8)$$

$$(Ww)(y) = \sum_{i=1}^{m} Q_i q_i(y) + \sum_{i=1}^{l} F_i f_i(y) + h(y), \quad (Ww)(y) = w(x) - d_\beta(y)w(\beta(y)). \quad (9)$$

Let us study the model in the space of Hölder class functions with weight:

$$H^\varsigma_J(x), \quad \rho(x) = x^{\varsigma_0}(1 - x)^{\varsigma_1}, \quad 0 < \varsigma < 1, \quad \varsigma < \varsigma_0 < 1 + \varsigma, \quad \varsigma < \varsigma_1 < 1 + \varsigma,$$

$$H_\vartheta_J(y), \quad \sigma(y) = y^{\vartheta_0}(1 - y)^{\vartheta_1}, \quad 0 < \vartheta < 1, \quad \vartheta < \vartheta_0 < 1 + \vartheta, \quad \vartheta < \vartheta_1 < 1 + \vartheta,$$

considering conditions of invertibility of operators $V$ and $W$ fulfilled

$$\theta_\alpha[1,c_\alpha(x),H_\varsigma_J(x,\rho)] \neq 0, \quad x \in J, \quad \theta_\beta[1,d_\beta(y),H_\vartheta_J(y,\sigma)] \neq 0, \quad y \in J. \quad (10)$$

From Theorem 1, inverse operators to operators $V$ and $W$ exist. Specific forms of inverse operators $V^{-1}$ and $W^{-1}$ can be found in [7]. Note that the series representing inverse operators in Lebesgue spaces with weight [4] are also suited for Hölder spaces with weight, when they converge.

For solving the system of equations, let us make use the approach for the solution of integral Fredholm equations of the second type with degenerate kernel [1].

First, let us apply operators $V^{-1}, W^{-1}$ to the left side of equations (8), (9); we obtain

$$v(x) = \sum_{i=1}^{n} P_i (V^{-1}p_i)(x) + \sum_{i=1}^{k} R_i (V^{-1}r_i)(x) + (V^{-1}g)(x), \quad (11)$$

$$w(y) = \sum_{i=1}^{m} Q_i (W^{-1}q_i)(y) + \sum_{i=1}^{l} F_i (W^{-1}f_i)(y) + (W^{-1}h)(y). \quad (12)$$
Having integrated the first equation of system (11) over intervals $[\nu_{j-1}, \nu_j]$, $j = 1, 2, ..., n$ corresponding to constants $P_j = \int_{\nu_{j-1}}^{\nu_j} v(x)dx$, and over intervals $[\epsilon_{j-1}, \epsilon_j]$, $j = 1, 2, ..., l$ corresponding to constants $E_j = \int_{\epsilon_{j-1}}^{\epsilon_j} v(x)dx$, and having subsequently integrated the second equation of system (12) over intervals $[\mu_{j-1}, \mu_j]$, $j = 1, 2, ..., m$ corresponding to constants $Q_j = \int_{\mu_{j-1}}^{\mu_j} w(y)dy$, and over intervals $[\gamma_{j-1}, \gamma_j]$, $j = 1, 2, ..., k$ corresponding to $R_j = \int_{\gamma_{j-1}}^{\gamma_j} w(y)dy$, we have a system of $r = n + m + l + k$ linear algebraic equations with the same number of unknowns.

Let us assume that the determinant of this system is different from zero $\det \Delta \neq 0$. After finding the unknown constants, we can define the solution of the balance system (11), (12). Thus, we have found the equilibrium state of the cyclic model of the system $S$, that is, we have obtained the state of the system to which it returns after the time $T$.

5 Results and conclusions

The theory of linear functional operators with shift is the adequate mathematical instrument for the investigation of systems with renewable resources [6, 2]. In this work, we present cyclic models for systems with two renewable resources. In modeling, the interactions and the reciprocal influences between the resources are taken into account. Balance correlations are found. Analysis of the models is carried out in weighted Hölder spaces. A method for the solution of balance equations, developed through the application of inverse operators to functional operators with shift, is offered. The equilibrium state of the system is obtained.

References


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