A Hopf Bifurcation of One Dimensional Attraction-Repulsion Chemotaxis System

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Abstract

This paper is concerned with an one dimensional attraction-repulsion chemotaxis system under homogeneous Neumann boundary conditions. We derive a free boundary problem of this system and examine the existence of stationary solutions and Hopf bifurcation which are essentially determined by the competition of attraction and repulsion.

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1. Introduction

We consider the following one-dimensional attraction-repulsion chemotaxis system ([6, 9, 10, 14]):

\[
\begin{align*}
\varepsilon \sigma \rho_t &= \varepsilon^2 \rho_{xx} - \varepsilon \kappa_1 (\rho a_x)_x + \varepsilon \kappa_2 (\rho b_x)_x + F(\rho, a), \quad x \in \Omega, \quad t > 0, \\
a_t &= a_{xx} + \mu \rho - a, \quad x \in \Omega, \quad t > 0, \\
b_t &= b_{xx} + \rho + a - b, \quad x \in \Omega, \quad t > 0,
\end{align*}
\]

(1)

where \( \rho(x, t) \) is the cell density, \( a(x, t) \) and \( b(x, t) \) are the chemical concentrations of attractant and repellent. The parameters \( \varepsilon, \sigma, \kappa_1, \kappa_2 \) and \( \mu \) are positive constants and \( H \) is a Heaviside step function. The nonlinear term \( F \) is given by \( F(\rho, a) = -\rho + H(\rho - a_0) - a \) with \( 0 < a_0 < 1/2 \).
The system with $\varepsilon = 1$ and $F = 0$ was first proposed in [13] to describe the quorum effect in chemotaxis and in [12] to describe the aggregation of microglia in Alzheimer’s disease. This system with $\Omega = (0, \infty)$ is globally well-posed in the sense that $\kappa_2 - \mu \kappa_1 > 0$ in [14]. The solution behavior of (1) with $F = 0$ in the multi-dimensional case was essentially determined by the competition of attraction and repulsion which is characterized by the sign of $\kappa_2 - \mu \kappa_1$ in [8, 15].

In this paper, we derive a free boundary problem of (1) when $\varepsilon = 0$ and find out the corresponding conditions in order to have the Hopf bifurcation of the free boundary problem in a finite interval $\Omega = (0, 1)$.

Hence, a free boundary problem of (1) when $\varepsilon = 0$ is given by:

$$
\begin{align*}
\frac{d\eta(t)}{dt} &= \frac{1}{\sigma}(C(a(\eta(t))) + \kappa_1 a_x(\eta(t), t) - \kappa_2 b_x(\eta(t), t)), \ x \in \eta(t), \quad (2)
\end{align*}
$$

where $C$ is a continuously differentiable function defined on an interval $I := (-a_0, 1-a_0)$, which is given by ([2, 7, 11])

$$
\begin{align*}
C(a(\eta)) &= -\frac{1 - 2a_0 - 2a(\eta)}{\sqrt{(a(\eta) + a_0)(1 - a_0 - a(\eta))}}. \quad (3)
\end{align*}
$$

Hence, a free boundary problem of (1) when $\varepsilon = 0$ is given by:

$$
\begin{align*}
a_t &= a_{xx} - (\mu + 1)a + \mu, \quad t > 0, \ x \in \Omega_1(t) \\
a_t &= a_{xx} - (\mu + 1)a, \quad t > 0, \ x \in \Omega_0(t) \\
a(\eta(t) - 0, t) &= a(\eta(t) + 0, t) \\
a_x(\eta(t) - 0, t) &= a_x(\eta(t) + 0, t)
\end{align*}
$$

and

$$
\begin{align*}
b_t &= b_{xx} - b + 1, \quad t > 0, \ x \in \Omega_1(t) \\
b_t &= b_{xx} - b, \quad t > 0, \ x \in \Omega_0(t) \\
b(\eta(t) - 0, t) &= b(\eta(t) + 0, t) \\
b_x(\eta(t) - 0, t) &= b_x(\eta(t) + 0, t). \quad (5)
\end{align*}
$$

The organization of the paper is as follows: In section 2, a change of variables is given which regularizes problem (4) and (5) in such a way that results from the theory of nonlinear evolution equations can be applied. In this way, we obtain a regularity of the solution which is sufficient for an analysis of the bifurcation. In section 3, we show the existence of equilibrium solutions for (4) and (5) and obtain the linearization of problem (4) and (5) under the condition $\kappa_2 - \mu \kappa_1 > 0$. In the last section, we investigate the conditions to obtain the periodic solutions and the bifurcation of the interface problem as the parameter $\sigma$ varies.
2. Regularization of the Interface Equation

Now, we consider the existence problem of (4) and (5).

\[
\begin{cases}
  a_t = a_{xx} - (\mu + 1)a + \mu H(x - \eta(t)), & 0 < x < 1, t > 0 \\
  b_t = b_{xx} - b + H(x - \eta(t)), & 0 < x < 1, t > 0 \\
  a_x(0, t) = 0, b_x(0, t) = 0, & t > 0 \\
  \sigma\eta'(t) = C(a(\eta)) + \kappa_1 a_x(\eta(t), t) - \kappa_2 b_x(\eta(t), t), & t > 0; \eta(0) = \eta_0.
\end{cases}
\]

(6)

Let \( A \) be an operator defined by \( A := -\frac{d^2}{dx^2} + \mu + 1 \) with domain \( D(A) = \{ a \in H^{2,2}((0, 1)) : a_x(0, t) = 0, a_x(1, t) = 0 \} \). Let \( A_0 := -\frac{d^2}{dx^2} + 1 \) with domain \( D(A_0) = \{ b \in H^{2,2}((0, 1)) : b_x(0, t) = 0, b_x(1, t) = 0 \} \). In order to apply semigroup theory to (6), we choose the space \( X := L_2(0, 1) \) with norm \( \| \cdot \|_2 \).

To get differential dependence on initial conditions, we decompose \( a \) in (6) into two parts: \( u \), which is a solution to a more regular problem and \( g \), which is less regular but explicitly known in terms of the Green’s function \( G \) of the operator \( A \). Namely, we define \( g : [0, 1] \times [0, 1] \to \mathbb{R} \) by

\[
g(x, \eta) := A^{-1}(\mu H(\cdot - \eta))(x) = \mu \int_0^1 G(x, y) H(y - \eta) dy,
\]

where \( G : [0, 1] \times [0, 1] \to \mathbb{R} \) is a Green’s function of \( A \) satisfying the Neumann boundary conditions, and \( \gamma : [0, 1] \to \mathbb{R} \),

\[
\gamma(\eta) := g(\eta, \eta).
\]

If we take a transformation \( u(t)(x) = a(x, t) - g(x, \eta(t)) \), we have \( (u_x)(t)(x) = a_x(x, t) - g_x(x, \eta(t)) \). Since \( G_x(x, \eta) \) is discontinuous, we cannot obtain one step more regular than that of (6).

To overcome this difficulty, let \( p(x, t) = a_x(x, t) \) and define \( \dot{g} : [0, 1] \times [0, 1] \to \mathbb{R} \),

\[
\dot{g}(x, \eta) := A^{-1}(\mu \delta(\cdot - \eta))(x) = \mu \int_0^1 \dot{G}(x, y) \delta(y - \eta) dy,
\]

where \( \dot{G} : [0, 1] \times [0, 1] \to \mathbb{R} \) is a Green’s function of \( A \) satisfying the Dirichlet boundary conditions, and \( \dot{\gamma} : [0, 1] \to \mathbb{R} \),

\[
\dot{\gamma}(\eta) := \dot{g}(\eta, \eta).
\]

We define \( j : [0, 1] \times [0, 1] \to \mathbb{R} \),

\[
j(x, \eta) := A_0^{-1}(H(\cdot - \eta))(r) = \int_0^1 J(x, y) H(y - \eta) dy
\]

and \( \alpha : [0, 1] \to \mathbb{R} \),

\[
\alpha(\eta) := j(\eta, \eta).
\]

Here \( J : [0, 1]^2 \to \mathbb{R} \) is a Green’s function of \( A_0 \) satisfying the boundary conditions. Define \( w(t)(x) = b(x, t) - j(x, \eta(t)) \), \( q(t)(x) = b_x(x, t) \) and define \( \dot{j} : [0, 1] \times [0, 1] \to \mathbb{R} \),

\[
\dot{j}(x, \eta) := A_0^{-1}(\mu \delta(\cdot - \eta))(x) = \mu \int_0^1 \dot{J}(x, y) \delta(y - \eta) dy
\]
Thus, we obtain an abstract evolution equation equivalent to (6) :

\[
\hat{J}(x, \eta) := A_0^{-1}(\delta(\cdot - \eta))(x) = \int_0^1 \hat{J}(x, y) \delta(y - \eta) \, dy,
\]

where \( \hat{J} : [0, 1] \times [0, 1] \to \mathbb{R} \) is a Green’s function of \( A_0 \) satisfying the Dirichlet boundary conditions and \( \hat{\alpha} : [0, 1] \to \mathbb{R} \),

\[
\hat{\alpha}(\eta) := \hat{J}(\eta, \eta).
\]

Applying the transformations \( u(t)(x) = a(x, t) - g(x, \eta(t)), v(t)(x) = p(x, t) - \hat{g}(x, \eta(t)) \) and \( w(t)(x) = b(x, t) - \hat{j}(x, \eta(t)) \), \( s(t)(x) = q(x, t) - \hat{j}(x, \eta(t)) \), then (6) becomes

\[
\begin{align*}
\dot{u} + Au &= \frac{1}{\sigma} \mu G(x, \eta)(C(u(\eta) + \gamma(\eta)) + \kappa_1(v(\eta) + \gamma(\eta)) - \kappa_2(s(\eta) + \hat{\alpha}(\eta))) \\
\dot{v} + Av &= -\frac{1}{\sigma} \hat{G}(x, \eta)(C(u(\eta) + \gamma(\eta)) + \kappa_1(v(\eta) + \gamma(\eta)) - \kappa_2(s(\eta) + \hat{\alpha}(\eta))) \\
\dot{w} + A_0 w &= \frac{1}{\sigma} J(x, \eta)(C(u(\eta) + \gamma(\eta)) + \kappa_1(v(\eta) + \gamma(\eta)) - \kappa_2(s(\eta) + \hat{\alpha}(\eta))) \\
\dot{s} + A_0 s &= -\frac{1}{\sigma} \hat{J}(x, \eta)(C(u(\eta) + \gamma(\eta)) + \kappa_1(v(\eta) + \gamma(\eta)) - \kappa_2(s(\eta) + \hat{\alpha}(\eta))) \\
\end{align*}
\]

\(
\eta'(t) = \frac{1}{\sigma}(C(u(\eta) + \gamma(\eta)) + \kappa_1(v(\eta) + \gamma(\eta)) - \kappa_2(s(\eta) + \hat{\alpha}(\eta))) , \ t > 0.
\)

Thus, we obtain an abstract evolution equation equivalent to (6) :

\[
\begin{align*}
\frac{d}{dt}(u, v, w, s, \eta) + \tilde{A}(u, v, w, s, \eta) &= \frac{1}{\sigma} f(u, v, w, s, \eta), \\
(u, v, w, s, \eta)(0) &= (u_0(x), v_0(x), w_0(x), s_0(x), \eta_0),
\end{align*}
\]

(8)

where \( \tilde{A} \) is a \( 5 \times 5 \) matrix where \( (1,1) \) and \( (2,2) \)-entries are an operator \( \hat{A}, (3,3) \) and \( (4,4) \)-entries are an operator \( A_0 \) and all the others are zero. The nonlinear forcing term \( f \) is

\[
f(u, v, w, s, \eta) = \begin{pmatrix}
f_1(\eta) \cdot (f_21(u, v, w, s, \eta) + f_22(u, v, w, s, \eta) - f_23(u, v, w, s, \eta)) \\
f_2(\eta) \cdot (f_21(u, v, w, s, \eta) + f_22(u, v, w, s, \eta) - f_23(u, v, w, s, \eta)) \\
f_3(\eta) \cdot (f_21(u, v, w, s, \eta) + f_22(u, v, w, s, \eta) - f_23(u, v, w, s, \eta)) \\
f_4(\eta) \cdot (f_21(u, v, w, s, \eta) + f_22(u, v, w, s, \eta) - f_23(u, v, w, s, \eta)) \\
f_21(u, v, w, s, \eta) + f_22(u, v, w, s, \eta) - f_23(u, v, w, s, \eta)
\end{pmatrix},
\]

where \( f_1 : (0, 1) \to X, \ f_1(\eta)(x) := \mu G(x, \eta), \ f_2 : (0, 1) \to X, \ f_2(\eta)(x) := -\frac{1}{\sigma} \hat{G}(x, \eta), \ f_3 : (0, 1) \to X, \ f_3(\eta)(x) := \hat{J}(x, \eta), \ f_4 : (0, 1) \to X, \ f_4(\eta)(x) := -\frac{1}{\sigma} \hat{J}(x, \eta), \ f_21 : W \to C, \ f_21(u, v, w, s, \eta) := C(u(\eta) + \gamma(\eta)), \ f_22 : W \to C, \ f_22(u, v, w, s, \eta) := C(u(\eta) + \gamma(\eta)), \end{pmatrix}

The well-posedness of solutions of (8) is shown in [5, 6, 16] with the help of the semigroup theory using domains of fractional powers \( \theta \in (3/4, 1] \) of \( A, A_0 \) and \( \tilde{A} \). Moreover, the nonlinear term \( f \) is a continuously differentiable function from \( W \cap \tilde{X}^\theta \).
to $\tilde{X}$, where $\tilde{X} := D(A) = D(A) \times D(A) \times D(A) \times D(A) \times \mathbb{R}$, $X^\theta := D(A^\theta) ; X^\theta_0 := D(A^\theta_0)$ and $\tilde{X}^\theta := D(A^\theta) = X^\theta \times X^\theta \times X^\theta_0 \times \mathbb{R}$.

The velocity of $\eta$ is denoted by

$$C(\eta) = \frac{1 - 2a_0 - 2(u(\eta) + \gamma(\eta))}{\sqrt{(u(\eta) + \gamma(\eta) + a_0)(1 - a_0 - (u(\eta) + \gamma(\eta)))}}.$$

The derivative of $f$ can be obtained from the following in [4]:

**Lemma 2.1.** The functions $G(\cdot, \eta) : (0,1) \to X, \hat{G}(\cdot, \eta) : (0,1) \to X, J(\cdot, \eta) : (0,1) \to X, \hat{J}(\cdot, \eta) : (0,1) \to X, C(\cdot) : W \to \mathbb{C}$ and $f : W \to X \times \mathbb{R}$ are continuously differentiable with derivatives given by

$$Df_{21}(u,v,w,s,\eta)(\hat{u}, \hat{v}, \hat{w}, \hat{s}, \hat{\eta}) = C'(u(\eta) + \gamma(\eta)) \cdot (u'(\eta)\hat{\eta} + \tilde{u}(\eta) + \gamma'(\eta)\hat{\eta})$$

$$Df_{22}(u,v,w,s,\eta)(\hat{u}, \hat{v}, \hat{w}, \hat{s}, \hat{\eta}) = \kappa_1(v'(\eta)\hat{\eta} + \tilde{v}(\eta) + \tilde{\gamma}'(\eta)\hat{\eta})$$

$$Df(u,v,w,s,\eta)(\hat{u}, \hat{v}, \hat{w}, \hat{s}, \hat{\eta}) = (f_{21}(u,v,w,s,\eta) + f_{22}(u,v,w,s,\eta)) \cdot (f_1'\eta, f_2'\eta, f_3'\eta, f_4'\eta, 0) \hat{\eta} + (Df_{21}(u,v,w,s,\eta) + Df_{22}(u,v,w,s,\eta))(\hat{u}, \hat{v}, \hat{w}, \hat{s}, \hat{\eta}) \cdot (f_1(\eta), f_2(\eta), f_3(\eta), f_4(\eta), 1).$$

3. **Equilibrium solutions and Linearization of the Interface equation**

In this section, we shall examine the existence of equilibrium solutions of (8). We look for $(u^*, v^*, w^*, s^*, \eta^*) \in D(A) \cap W$ satisfying the following equations:

$$\begin{aligned}
Au &= \frac{1}{\sigma} \mu G(\cdot, \eta)(C(u(\eta) + \gamma(\eta)) + \kappa_1(v(\eta) + \hat{\gamma}(\eta))) - \kappa_2(s(\eta) + \hat{\alpha}(\eta))) \\
Av &= -\frac{1}{\sigma_1} \mu \hat{G}(\cdot, \eta)(C(u(\eta) + \gamma(\eta)) + \kappa_1(v(\eta) + \hat{\gamma}(\eta))) - \kappa_2(s(\eta) + \hat{\alpha}(\eta))) \\
A_0w &= \frac{1}{\sigma} J(\cdot, \eta^*)(C(u(\eta) + \gamma(\eta)) + \kappa_1(v(\eta) + \hat{\gamma}(\eta))) - \kappa_2(s(\eta) + \hat{\alpha}(\eta))) \\
A_0s &= -\frac{1}{\sigma_1} \hat{J}(\cdot, \eta^*)(C(u(\eta) + \gamma(\eta)) + \kappa_1(v(\eta) + \hat{\gamma}(\eta))) - \kappa_2(s(\eta) + \hat{\alpha}(\eta))) \\
0 &= C(u(\eta) + \gamma(\eta)) + \kappa_1(v(\eta) + \hat{\gamma}(\eta)) - \kappa_2(s(\eta) + \hat{\alpha}(\eta))) \\
u'(0) = 0 = u'(1), v(0) = 0 = v(1), w'(0) = 0 = w'(1), s(0) = 0 = s(1).
\end{aligned}$$

**Theorem 3.1.** Suppose that $\frac{\mu \cosh \frac{\sqrt{1+\mu}}{(1+\mu) \sinh \frac{\sqrt{1+\mu}}{2}}}{\sqrt{1+\mu}} < \frac{1}{2} - a_0 < \frac{\mu}{1+\mu}$ and $\mu \kappa_1 < \kappa_2$.

Then equation (8) has at least one equilibrium solution $(0,0,0,\eta^*), \eta^* \in (0,1/3)$. The linearization of $f$ at the stationary solution $(0,0,0,\eta^*)$ is

$$Df(0,0,0,\eta^*)(\hat{u}, \hat{v}, \hat{w}, \hat{s}, \hat{\eta}) = \begin{pmatrix}
\mu G(\cdot, \eta^*) Q(\hat{u}, \hat{v}, \hat{w}, \hat{s}, \hat{\eta}) \\
-\frac{\mu}{\eta^2} \hat{G}(\cdot, \eta^*) Q(\hat{u}, \hat{v}, \hat{w}, \hat{s}, \hat{\eta}) \\
J(\cdot, \eta^*) Q(\hat{u}, \hat{v}, \hat{w}, \hat{s}, \hat{\eta}) \\
-\frac{1}{\eta^2} \hat{J}(\cdot, \eta^*) Q(\hat{u}, \hat{v}, \hat{w}, \hat{s}, \hat{\eta}) \\
Q(\hat{u}, \hat{v}, \hat{w}, \hat{s}, \hat{\eta})
\end{pmatrix},$$
where \( Q(\hat{u}, \hat{v}, \hat{w}, \hat{\eta}) = 4(\hat{u}(\eta^*) + \gamma'(\eta^*)\hat{\eta}) + \kappa_1(\hat{v}(\eta^*) + \hat{\gamma}'(\eta^*)\hat{\eta}) - \kappa_2(\hat{s}(\eta^*) + \hat{\alpha}'(\eta^*)\hat{\eta}) \).

The pair \((0, 0, 0, \eta^*)\) corresponds to a unique steady state \((a^*, p^*, b^*, q^*, \eta^*)\) of (6) for \( \sigma \neq 0 \) with \( a^*(x) = g(x, \eta^*), p^*(x) = \hat{g}(x, \eta^*), b^*(x) = j(x, \eta^*) \) and \( q^*(x) = \hat{j}(x, \eta^*) \).

**Proof:** From the system of equations (10), we have \( u^* = 0, v^* = 0, w^* = 0 \) and \( s^* = 0 \). In order to show existence of \( \eta^* \), we define

\[
\Gamma(\eta) := C(\gamma(\eta)) + \kappa_1 \hat{\gamma}(\eta) - \kappa_2 \hat{\alpha}(\eta).
\]

We shall show that \( \Gamma'(\eta) < 0 \) for all \( 0 < \eta < \frac{1}{3} \) and then \( \Gamma(\eta) = 0 \) is solvable with \( \eta^* \) if \( \Gamma(0) > 0 \) and \( \Gamma(1/3) < 0 \) for all \( 0 < \eta < \frac{1}{3} \).

Suppose that \( \kappa_2 > \mu \kappa_1 \). Since \( \cosh(\sqrt{1 + \mu}(1 - 2\eta)) - \cosh(\sqrt{1 + \mu} \eta) > \cosh(1 - 2\eta) - \cosh \eta \) for all \( 0 < \eta < \frac{1}{3} \),

\[
\Gamma'(\eta) = C'(\gamma(\eta)) \gamma'(\eta) + \kappa_1 \hat{\gamma}'(\eta) - \kappa_2 \hat{\alpha}'(\eta) < C'(\gamma(\eta)) \gamma'(\eta) + \kappa_2 \Delta(\mu, \eta),
\]

where

\[
\Delta(\mu, \eta) := \frac{1}{\mu} \hat{\gamma}'(\eta) - \hat{\alpha}'(\eta) = \frac{\cosh(\sqrt{1 + \mu}(1 - 2\eta)) - \cosh(\sqrt{1 + \mu} \eta)}{\sqrt{1 + \mu} \sinh \sqrt{1 + \mu}} - \frac{\cosh(1 - 2\eta) - \cosh \eta}{\sinh 1}.
\]

We have \( \Delta(\mu, 0) = \frac{1}{\sqrt{1 + \mu}} \tanh \frac{\sqrt{1 + \mu}}{2} - \tanh \frac{1}{2} < 0 \) and \( \Delta(\mu, \frac{1}{3}) = 0 \).

\[
\frac{\partial}{\partial \eta} \Delta(\mu, \eta) = -\frac{2 \sinh(\sqrt{1 + \mu}(1 - 2\eta)) - \sinh(\sqrt{1 + \mu} \eta)}{\sqrt{1 + \mu} \sinh(\sqrt{1 + \mu})} + \frac{2 \sinh(1 - 2\eta) + \sinh \eta}{\sinh 1} = \frac{2 \sinh(1 - 2\eta) - \sinh(\sqrt{1 + \mu} \eta)}{\sinh \eta} = \frac{\sinh \sqrt{1 + \mu} \phi(1 - 2\eta) + \sinh \eta \phi(\eta)}{\sinh \eta},
\]

where \( \phi(\eta) = \frac{\sinh \sqrt{1 + \mu}}{\sinh 1} - \frac{\sinh(\sqrt{1 + \mu} \eta)}{\sinh \eta}. \) We note that \( \phi(\eta) \) and \( \phi(1 - 2\eta) \) are positive for all \( 0 < \eta < 1 \) since \( \phi(1) = 0, \phi(0) = \frac{\sinh \sqrt{1 + \mu}}{\sinh 1} - \sqrt{1 + \mu} > 0 \) and

\[
\phi'(\eta) = \left( -\frac{1}{\sqrt{1 + \mu}} \tanh \eta + \tanh \sqrt{1 + \mu} \eta \right) \frac{\cosh \eta \cosh(\sqrt{1 + \mu} \eta)}{(\sinh \eta)^2} < 0
\]

for all \( 0 < \eta < 1 \). Hence \( \frac{\partial}{\partial \eta} \Delta(\mu, \eta) > 0 \) for \( 0 < \eta < \frac{1}{3} \) and so, \( \Delta(\mu, \eta) < 0 \) for \( 0 < \eta < \frac{1}{3} \) and thus \( \Gamma'(\eta) \) is negative for \( 0 < \eta < \frac{1}{3} \) if \( \mu \kappa_1 < \kappa_2 \). Since \( \hat{\gamma}(1/3) = 0 \) and \( \hat{\alpha}(1/3) = 0, \Gamma(1/3) < 0 \) implies that \( \gamma(1/3) < 1/2 - a_0 \). Hence \( \eta^* \) exists if \( \gamma(1/3) < 1/2 - a_0 < \gamma(0) \) with \( \mu \kappa_1 < \kappa_2 \).

The formula for \( Df(0, 0, 0, 0, \eta^*) \) follows from the relation \( C'(1/2 - a_0) = 4, \) and the corresponding steady state \((a^*, p^*, b^*, q^*, \eta^*)\) for (6) is obtained by using the transformation and Theorem 2.1 in [4].

**4. A Hopf bifurcation**

In this section, we shall show that there exists a Hopf bifurcation from the curve \( \sigma \mapsto (0, 0, 0, 0, \eta^*) \) of the equilibrium solution. First, let us introduce the following relevant definition.

**Definition 4.1.** Under the assumptions of Theorem 3.1, define (for \( 1 \geq \theta > 3/4 \)) the linear operator \( B \) from \( \hat{X}^\theta \) to \( \hat{X} \) by

\[
B := Df(0, 0, 0, 0, \eta^*).
\]
We then define \( (0, 0, 0, 0, \eta^*) \) to be a Hopf point for (8) if and only if there exists an \( \epsilon_0 > 0 \) and a \( C^1 \)-curve
\[
(-\epsilon_0 + \tau^*, \tau^* + \epsilon_0) \mapsto (\lambda(\tau), \phi(\tau)) \in \mathbb{C} \times \tilde{X}_C
\]
where \( \tilde{X}_C \) denotes the complexification of the real space \( Y \) of eigendata for \(-\tilde{A} + \tau B\) with

(i) \( (-\tilde{A} + \tau B)(\phi(\tau)) = \lambda(\tau)\phi(\tau), \quad (-\tilde{A} + \tau B)(\overline{\phi(\tau)}) = \overline{\lambda(\tau)\overline{\phi(\tau)}} \);

(ii) \( \text{Re}(\lambda) = \beta i \) with \( \beta > 0 \);

(iii) \( \text{Re}(\lambda) \neq 0 \) for all \( \lambda \) in the spectrum of \((-\tilde{A} + \tau^* B) \setminus \{ \pm i\beta \} \);

(iv) \( \text{Re}\chi(\tau^*) \neq 0 \) (transversality);

where \( \tau = 1/\sigma \).

Next, we check (8) for Hopf points. For this, we solve the eigenvalue problem:
\[
-\tilde{A}(u, v, w, s, \eta) + \tau B(u, v, w, s, \eta) = \lambda I_5(u, v, w, s, \eta),
\]
where \( I_5 \) is an \( 5 \times 5 \) identity matrix. This is equivalent to:
\[
\begin{cases}
(A + \lambda)u = \tau \mu G(\cdot, \eta^*)(4(u(\eta^*) + \gamma'(\eta^*)\eta) + \kappa_1(v(\eta^*) + \gamma'(\eta^*)\eta) - \kappa_2(s(\eta^*) + \alpha'(\eta^*)\eta)) \\
(A + \lambda)v = -\frac{\tau \mu \hat{G}(\cdot, \eta^*)}{\eta^*}(4(u(\eta^*) + \gamma'(\eta^*)\eta) + \kappa_1(v(\eta^*) + \gamma'(\eta^*)\eta) - \kappa_2(s(\eta^*) + \alpha'(\eta^*)\eta)) \\
(A_0 + \lambda)w = \tau J(\cdot, \eta^*)(4(u(\eta^*) + \gamma'(\eta^*)\eta) + \kappa_1(v(\eta^*) + \gamma'(\eta^*)\eta) - \kappa_2(s(\eta^*) + \alpha'(\eta^*)\eta)) \\
(A_0 + \lambda)s = -\frac{\tau J(\cdot, \eta^*)}{\eta^*}(4(u(\eta^*) + \gamma'(\eta^*)\eta) + \kappa_1(v(\eta^*) + \gamma'(\eta^*)\eta) - \kappa_2(s(\eta^*) + \alpha'(\eta^*)\eta)) \\
\lambda = \tau(4(u(\eta^*) + \gamma'(\eta^*)\eta) + \kappa_1(v(\eta^*) + \gamma'(\eta^*)\eta) - \kappa_2(s(\eta^*) + \alpha'(\eta^*)\eta)).
\end{cases}
\tag{11}
\]

We shall show that an equilibrium solution is a Hopf point.

**Theorem 4.2.** Suppose that \( \frac{\mu \cosh \frac{\sqrt{\mu \eta^*}}{1+\mu}}{(1+\mu) \sinh \sqrt{\mu \eta^*}} < \frac{1}{2} - a_0 < \frac{\mu}{1+\mu} \) and \( \mu \kappa_1 < \kappa_2 \). Moreover, assume that \( \kappa_1 \) satisfies that \( \frac{\kappa_1}{\eta^*} < 4 \). Additionally, suppose the operator \(-\tilde{A} + \tau^* B\) has a unique pair \( \{ \pm i\beta \} \), \( \beta > 0 \) of purely imaginary eigenvalues for some \( \tau^* > 0 \). Then, \( (0, 0, 0, 0, \eta^*, \tau^*) \) is a Hopf point for (8).

**Proof:** We assume without loss of generality that \( \beta > 0 \), and \( \Phi^* \) is the (normalized) eigenfunction of \(-\tilde{A} + \tau^* B\) with eigenvalue \( i\beta \). We have to show that \( (\Phi^*, i\beta) \) can be extended to a \( C^1 \)-curve \( \tau \mapsto (\Phi(\tau), \lambda(\tau)) \) of eigendata for \(-\tilde{A} + \tau B\) with \( \text{Re}(\chi(\tau^*)) \neq 0 \).

For this, let \( \Phi^* = (\psi_0, v_0, w_0, s_0, \eta_0) \in D(A) \times D(A) \times D(A_0) \times D(A_0) \times \mathbb{R} \). First, we note that \( \eta_0 \neq 0 \). Otherwise, by (11), \( (A + i\beta)\psi_0 = \mu i\beta \eta_0 G(\cdot, \eta^*) = 0 \) and \( (A + i\beta)v_0 = -\frac{\mu}{\eta^*} i\beta \eta_0 \hat{G}(\cdot, \eta^*) = 0 \), which is not possible given \( A \) is symmetric. So, without loss of generality, let \( \eta_0 = 1 \). Then \( E(\psi_0, v_0, w_0, s_0, i\beta, \tau^*) = 0 \) by (11), where
\[
E : D(A) \times D(A) \times D(A_0) \times D(A_0) \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C},
\]
\[ E(u, v, w, s, \lambda, \tau) := \\
(A + \lambda)u - \tau \mu G(\cdot, \eta^*)(4(u(\eta^*) + \gamma'(\eta^*)) + \kappa_1(v(\eta^*) + \tilde{\gamma}'(\eta^*)) - \kappa_2(s(\eta^*) + \tilde{\alpha}'(\eta^*))) \\
(A + \lambda)v + \tau \frac{\partial}{\partial \eta} \tilde{G}(\cdot, \eta^*)(4(u(\eta^*) + \gamma'(\eta^*)) + \kappa_1(v(\eta^*) + \tilde{\gamma}'(\eta^*)) - \kappa_2(s(\eta^*) + \tilde{\alpha}'(\eta^*))) \\
(A_0 + \lambda)w - \tau J(\cdot, \eta^*)(4(u(\eta^*) + \alpha'(\eta^*)) + \kappa_1(v(\eta^*) + \gamma'(\eta^*)) - \kappa_2(s(\eta^*) + \alpha'(\eta^*))) \\
(A_0 + \lambda)s + \tau \frac{1}{\eta^*} \tilde{J}(\cdot, \eta^*)(4(u(\eta^*) + \gamma'(\eta^*)) + \kappa_1(v(\eta^*) + \tilde{\gamma}'(\eta^*)) - \kappa_2(s(\eta^*) + \tilde{\alpha}'(\eta^*)) \\
\lambda - \tau(4(u(\eta^*) + \gamma'(\eta^*)) + \kappa_1(v(\eta^*) + \tilde{\gamma}'(\eta^*)) - \kappa_2(s(\eta^*) + \tilde{\alpha}'(\eta^*)) \\
\lambda - \tau(4(u(\eta^*) + \gamma'(\eta^*)) + \kappa_1(v(\eta^*) + \tilde{\gamma}'(\eta^*)) - \kappa_2(s(\eta^*) + \tilde{\alpha}'(\eta^*)) \\
\lambda - \tau(4(u(\eta^*) + \gamma'(\eta^*)) + \kappa_1(v(\eta^*) + \tilde{\gamma}'(\eta^*)) - \kappa_2(s(\eta^*) + \tilde{\alpha}'(\eta^*))
\]

The equation \( E(u, v, w, s, \lambda, \tau) = 0 \) is equivalent to \( \lambda \) being an eigenvalue of \(-\tilde{A} + \tau B\) with eigenfunction \((u, v, w, s, 1)\). We shall apply the implicit function theorem to \( E \) to check that \( E \) is of \( C^1 \)— class and that

\[ D_{(u,v,w,s,\lambda)}E(\psi_0, v_0, w_0, s_0, i\beta, \tau^*) \in L((D(A)_C)^2 \times (D(A_0)_C)^2 \times \mathbb{C} \times \mathbb{R}, X_C^4 \times \mathbb{C}) \]  

is an isomorphism. In addition, the mapping

\[ D_{(u,v,w,s,\lambda)}E(\psi_0, v_0, w_0, s_0, i\beta, \tau^*)(\hat{u}, \hat{v}, \hat{w}, \hat{s}, \hat{\lambda}) \]

\[ = \begin{pmatrix}
(A + i\beta)\hat{u} + \hat{\lambda}\psi_0 - \tau^* \mu G(\cdot, \eta^*)(4\hat{u}(\eta^*) + \kappa_1\hat{v}(\eta^*) - \kappa_2\hat{s}(\eta^*)) \\
(A + i\beta)\hat{v} + \hat{\lambda}\psi_0 - \tau^* \frac{\partial}{\partial \eta} \tilde{G}(\cdot, \eta^*)(4\hat{u}(\eta^*) + \kappa_1\hat{v}(\eta^*) - \kappa_2\hat{s}(\eta^*)) \\
(A_0 + i\beta)\hat{w} + \hat{\lambda}\psi_0 - \tau^* J(\cdot, \eta^*)(4\hat{u}(\eta^*) + \kappa_1\hat{v}(\eta^*) - \kappa_2\hat{s}(\eta^*)) \\
(A_0 + i\beta)\hat{s} + \hat{\lambda}\psi_0 - \tau^* \frac{1}{\eta^*} \tilde{J}(\cdot, \eta^*)(4\hat{u}(\eta^*) + \kappa_1\hat{v}(\eta^*) - \kappa_2\hat{s}(\eta^*)) \\
\hat{\lambda} - \tau^* (4\hat{u}(\eta^*) + \kappa_1\hat{v}(\eta^*) - \kappa_2\hat{s}(\eta^*)) \\
\end{pmatrix} \]

is a compact perturbation of the mapping

\[ (\hat{u}, \hat{v}, \hat{w}, \hat{s}, \hat{\lambda}) \rightarrow ((A + i\beta)\hat{u}, (A + i\beta)\hat{v}, (A_0 + i\beta)\hat{w}, (A_0 + i\beta)\hat{s}, \hat{\lambda}) \]

which is invertible. Thus, \( D_{(u,v,w,s,\lambda)}E(\psi_0, v_0, w_0, s_0, i\beta, \tau^*) \) is a Fredholm operator of index 0. Therefore, in order to verify (12), it suffices to show that the system of equations

\[ D_{(u,v,w,s,\lambda)}E(\psi_0, v_0, w_0, s_0, i\beta, \tau^*)(\hat{u}, \hat{v}, \hat{w}, \hat{s}, \hat{\lambda}) = 0 \]  

necessarily implies that \( \hat{u} = 0, \hat{v} = 0, \hat{w} = 0, \hat{s} = 0 \) and \( \hat{\lambda} = 0 \). If we define \( \phi := \psi_0 - \mu G(\cdot, \eta^*), \xi := v_0 + \frac{\mu}{\eta^*} \tilde{G}(\cdot, \eta^*), \rho = w_0 - J(\cdot, \eta^*) \) and \( \zeta = s_0 + \frac{1}{\eta^*} \tilde{J}(\cdot, \eta^*) \), then (13) becomes

\[ (A + i\beta)\hat{u} + \hat{\lambda}\phi = 0, \]  

(14)
\[ (A + i\beta)\hat{v} + \hat{\lambda}\xi = 0, \]  

(15)
\[ (A_0 + i\beta)\hat{w} + \hat{\lambda}\rho = 0, \]  

(16)
\[ (A_0 + i\beta)\hat{s} + \hat{\lambda}\zeta = 0, \]  

(17)
\[ \frac{\hat{\lambda}}{\tau^*} = 4\hat{u}(\eta^*) + \kappa_1\hat{v}(\eta^*) - \kappa_2\hat{s}(\eta^*). \]  

(18)
Thus (26) implies that
\[ \int \left( \frac{\partial}{\partial \tau} \right) = 0, \]
\[ \phi, \xi, \rho \quad \text{and} \quad \zeta \]
are solutions to the equations, we have:
\begin{align*}
(A + i \beta)\phi &= -\mu \delta \eta, \quad (19) \\
(A + i \beta)\xi &= \mu \eta^* \delta \eta, \quad (20) \\
(A_0 + i \beta)\rho &= -\delta \eta, \quad (21) \\
(A_0 + i \beta)\zeta &= \frac{1}{\eta^*} \delta \eta, \quad (22)
\end{align*}

\[ \frac{\partial}{\partial \tau} = 4(\dot{\phi}(\eta^*) + \mu G(\eta^*, \eta^*) + \gamma'(\eta^*)) + \kappa_1(\xi(\eta^*) - \frac{\mu}{\eta} \dot{G}(\eta^*, \eta^*) + \gamma'(\eta^*)) \\
- \kappa_2(\zeta(\eta^*) - \frac{1}{\eta^*} \dot{J}(\eta^*, \eta^*) + \alpha'(\eta^*)). \quad (23)
\]

Multiplying (15) and (20) by \( \phi \), and (14) and (19) by \( \xi \) and subtracting one from the other, we now obtain
\begin{align*}
\dot{u}(\eta^*) &= -\eta^* \dot{v}(\eta^*), \quad \dot{v}(\eta^*) = -\eta^* \dot{s}(\eta^*) \quad (24) \\
\phi(\eta^*) &= -\eta^* \xi(\eta^*), \quad \rho(\eta^*) = -\eta^* \zeta(\eta^*). \quad (25)
\end{align*}

Multiplying (14) by \( 4 \dot{\phi}, \) (15) by \( -\eta^* \kappa_1 \xi \), and (17) by \( \mu \eta^* \kappa_2 \zeta \) and adding the resultants to each, we obtain
\begin{align*}
0 &= -4 \mu \dot{u}(\eta^*) - \kappa_1 \mu \dot{v}(\eta^*) + \mu \kappa_2 \dot{s}(\eta^*) + \lambda(4||\phi||^2 - \eta^* \kappa_1 ||\xi||^2 + \mu \eta^* \kappa_2 ||\zeta||^2) \\
&\quad + 2i \beta \int \left( 4 \dot{u} \xi - \eta^* \kappa_1 \dot{v} \xi + \mu \eta^* \kappa_2 \dot{s} \xi \right). \quad (26)
\end{align*}

Multiplying (19) by \( 4 \dot{\phi}, \) (20) by \( -\eta^* \kappa_1 \xi \) and (17) by \( \mu \eta^* \kappa_2 \zeta \) and adding the resultants to each, we obtain
\begin{align*}
4||A^{1/2} \phi||^2 - \eta^* \kappa_1 ||A^{1/2} \xi||^2 + \mu \eta^* \kappa_2 ||A_0^{1/2} \zeta||^2 \\
&+ i \beta(4||\phi||^2 - \eta^* \kappa_1 ||\xi||^2 + \mu \eta^* \kappa_2 ||\zeta||^2) \\
&= -4 \mu \dot{\phi}(\eta^*) - \mu \kappa_1 \xi(\eta^*) + \mu \eta^* \kappa_2 \zeta(\eta^*),
\end{align*}

and from (23), we get
\[ \frac{\mu}{\tau^*} = 4||\phi||^2 - \eta^* \kappa_1 ||\xi||^2 + \mu \eta^* \kappa_2 ||\zeta||^2. \quad (27)\]

Thus (26) implies that
\[ \int \left( 4 \dot{u} \xi - \eta^* \kappa_1 \dot{v} \xi + \mu \eta^* \kappa_2 \dot{s} \xi \right) = 0. \quad (28)\]

Now, multiplying (14) by \( 4 \dot{u}, \) (19) by \( -\eta^* \kappa_1 \dot{v} \) and (17) by \( \mu \eta^* \kappa_2 \dot{s} \) and adding the resultants to each, we obtain
\begin{align*}
4||A^{1/2} \dot{u}||^2 - \eta^* \kappa_1 ||A^{1/2} \dot{v}||^2 + \mu \eta^* \kappa_2 ||A_0^{1/2} \dot{s}||^2 \\
&+ \lambda \int \left( 4 \dot{\phi} \xi - \eta^* \kappa_1 \dot{v} \xi + \mu \eta^* \kappa_2 \dot{s} \xi \right) = 0
\end{align*}
and from (28), we have
\[ \left\{ \begin{align*}
4||A^{1/2} \dot{u}||^2 - \eta^* \kappa_1 ||A^{1/2} \dot{v}||^2 + \mu \eta^* \kappa_2 ||A_0^{1/2} \dot{s}||^2 &= 0 \\
4||\dot{u}||^2 - \eta^* \kappa_1 ||\dot{v}||^2 + \mu \eta^* \kappa_2 ||\dot{s}||^2 &= 0.
\end{align*} \quad (29)\]
and applying (24) to the above equation, we have
\[ \|A^{1/2}\phi\|^2 + i\beta\|\phi\|^2 = -\mu\phi\) and \(\|A^{1/2}\xi\|^2 + i\beta\|\xi\|^2 = \frac{\mu}{\eta^*}\xi\omega\)\]
and applying (24) to the above equation, we have
\[ \|A^{1/2}\phi\|^2 = (\eta^*)^2\|A^{1/2}\xi\|^2 and \|\phi\|^2 = (\eta^*)^2\|\xi\|^2. \]  
(30)

Now, multiplying (14) by \(2i\beta\) and (19) by \(\tilde{X}\) and subtracting the resultants from each, we now obtain
\[ 2i\beta(\|A^{1/2}\hat{u}\|^2 - (\eta^*)^2\|A^{1/2}\hat{v}\|^2) - 2\beta^2(\|\hat{u}\|^2 - (\eta^*)^2\|\hat{v}\|^2) + \hat{\lambda}(\|\phi\|^2 - (\eta^*)^2\|\xi\|^2). \]
Applying (30) to the above equation, we have
\[ \|A^{1/2}\hat{u}\|^2 - (\eta^*)^2\|A^{1/2}\hat{v}\|^2 = 0 and \|\hat{u}\|^2 - (\eta^*)^2\|\hat{v}\|^2 = 0 \]
and thus (29) implies:
\[ (4 - \frac{\kappa_1}{\eta^*})\|\hat{u}\|^2 + \mu\kappa_2\eta^*\|\hat{s}\|^2 = 0. \]

Since \(4 - \frac{\kappa_1}{\eta^*} > 0\), we have \(\hat{u} = 0\) and \(\hat{s} = 0\) and so, \(\hat{v} = 0\). By (16) and (18), we have \(\hat{w} = 0\) and \(\hat{\lambda} = 0\). □

**Theorem 4.3.** Under the same condition as in Theorem 4.2, \((0, 0, 0, 0, \eta^*, \tau^*)\) satisfies the transversality condition. Hence, it is a Hopf point for (8).

**Proof:** By implicit differentiation of \(E(\psi_0(\tau), v_0(\tau), w_0(\tau), s_0(\tau), \lambda(\tau), \tau) = 0\), we find
\[ D_{(u,v,w,s,\lambda)}E(\psi_0, v_0, w_0, s_0, i\beta, \tau^*)(\psi'_0(\tau^*), v'_0(\tau^*), w'_0(\tau^*), s'_0(\tau^*), \lambda'(\tau^*)) = \]
\[ \begin{pmatrix}
\mu G(\cdot, \eta^*)(4\psi_0(\eta^*) + \gamma'(\eta^*)) + \kappa_1(v_0(\eta^*) + \gamma'(\eta^*)) - \kappa_2(s_0(\eta^*) + \gamma'(\eta^*)) \\
-\frac{\mu}{\eta^*} \hat{G}(\cdot, \eta^*)(4\psi_0(\eta^*) + \gamma'(\eta^*)) + \kappa_1(v_0(\eta^*) + \gamma'(\eta^*)) - \kappa_2(s_0(\eta^*) + \gamma'(\eta^*)) \\
\hat{J}(\cdot, \eta^*)(4\psi_0(\eta^*) + \gamma'(\eta^*)) + \kappa_1(v_0(\eta^*) + \gamma'(\eta^*)) - \kappa_2(s_0(\eta^*) + \gamma'(\eta^*)) \\
-\frac{1}{\eta^*} \hat{J}(\cdot, \eta^*)(4\psi_0(\eta^*) + \gamma'(\eta^*)) + \kappa_1(v_0(\eta^*) + \gamma'(\eta^*)) - \kappa_2(s_0(\eta^*) + \gamma'(\eta^*))
\end{pmatrix}. \]

This means that the functions \(\hat{u} := \psi'_0(\tau^*), \hat{v} := v'_0(\tau^*), \hat{w} := w'_0(\tau^*), \hat{s} := s'_0(\tau^*)\) and \(\hat{\lambda} := \lambda'(\tau^*)\) satisfy the equations
\[ \begin{pmatrix}
(A + i\beta)\hat{u} + \hat{\lambda}\psi_0 - \tau^*\mu G(\cdot, \eta^*)(4\hat{u}(\eta^*) + \kappa_1\hat{v}(\eta^*) - \kappa_2\hat{s}(\eta^*)) = \mu G(\cdot, \eta^*)Z(\eta^*) \\
(A + i\beta)\hat{v} + \hat{\lambda}\xi_0 + \tau^*\mu \hat{G}(\cdot, \eta^*)(4\hat{u}(\eta^*) + \kappa_1\hat{v}(\eta^*) - \kappa_2\hat{s}(\eta^*)) = -\frac{\mu}{\eta^*} \hat{G}(\cdot, \eta^*)Z(\eta^*) \\
(A_0 + i\beta)\hat{w} + \hat{\lambda}\rho_0 - \tau^*\hat{J}(\cdot, \eta^*)(4\hat{u}(\eta^*) + \kappa_1\hat{v}(\eta^*) - \kappa_2\hat{s}(\eta^*)) = \hat{J}(\cdot, \eta^*)Z(\eta^*) \\
(A_0 + i\beta)\hat{s} + \hat{\lambda}\zeta_0 + \tau^*\hat{J}(\cdot, \eta^*)(4\hat{u}(\eta^*) + \kappa_1\hat{v}(\eta^*) - \kappa_2\hat{s}(\eta^*)) = -\hat{J}(\cdot, \eta^*)Z(\eta^*) \\
\hat{\lambda} - \tau^*(4\hat{u}(\eta^*) + \kappa_1\hat{v}(\eta^*) - \kappa_2\hat{s}(\eta^*)) = Z(\eta^*)
\end{pmatrix}, \]

(31)
where $Z(\eta^*) = 4(\psi_0(\eta^*) + \gamma'(\eta^*)) + \kappa_1(v_0(\eta^*) + \dot{\gamma}'(\eta^*)) - \kappa_2(s_0(\eta^*) + \dot{\alpha}'(\eta^*))$.

By letting $\phi := \psi_0 - \mu G(\cdot, \eta^*)$, $\xi = v_0 + \frac{\mu}{\eta^*} \dot{G}(\cdot, \eta^*)$, $\rho = w_0 - J(\cdot, \eta^*)$ and $\zeta = s_0 - J(\cdot, \eta^*)$ as before, we obtain

\[
(A + i\beta)\bar{u} + \bar{\lambda}\phi = 0, \tag{32}
\]

\[
(A + i\beta)\bar{v} + \bar{\lambda}\xi = 0, \tag{33}
\]

\[
(A_0 + i\beta)\bar{w} + \bar{\lambda}\rho = 0, \tag{34}
\]

\[
(A_0 + i\beta)\bar{s} + \bar{\lambda}\zeta = 0, \tag{35}
\]

\[
\bar{\lambda} - \tau^*(4\tilde{u}(\eta^*) + \kappa_1\tilde{v}(\eta^*) - \kappa_2\tilde{s}(\eta^*)) = \frac{i\beta}{\eta^*}. \tag{36}
\]

Multiplying (32) by $4\bar{\phi}$, (33) by $-\eta^*\kappa_1\bar{\xi}$ and (34) by $\mu\eta^*\kappa_2\bar{\zeta}$ and adding the resultants to each, we now obtain

\[
-4\mu\tilde{u}(\eta^*) - \kappa_1\mu\tilde{v}(\eta^*) + \mu\eta^*\kappa_2\tilde{s}(\eta^*) + \bar{\lambda}(4||\phi||^2 - \eta^*\kappa_1||\xi||^2 + \mu\eta^*\kappa_2||\zeta||^2)
+ 2i\beta \int (4\tilde{u}\phi - \eta^*\kappa_1\tilde{v}\bar{\xi} + \mu\eta^*\kappa_2\tilde{s}\bar{\zeta}) = 0.
\]

From (36), the above equation implies that

\[
\frac{i\beta\mu}{(\tau^*)^2} + 2i\beta \int (4\tilde{u}\phi - \eta^*\kappa_1\tilde{v}\bar{\xi} + \mu\eta^*\kappa_2\tilde{s}\bar{\zeta}) = 0. \tag{37}
\]

Multiplying (32) by $4\bar{u}$, (33) by $-\eta^*\kappa_1\bar{\xi}$ and (35) by $\mu\eta^*\kappa_2\bar{\zeta}$ and summing their resultants to each, and then apply (37) then we now obtain

\[
4||A^{1/2}\bar{u}||^2 - \eta^*\kappa_1||A^{1/2}\bar{v}||^2 + \mu\eta^*\kappa_2||A_0^{1/2}\bar{s}||^2 + i\beta(4||\bar{u}||^2 - \eta^*\kappa_1||\bar{v}||^2 + \mu\eta^*\kappa_2||\bar{s}||^2)
= \frac{\mu}{2(\tau^*)^2} \bar{\lambda}.
\]

The real part of the above is given by

\[
4||A^{1/2}\bar{u}||^2 - \eta^*\kappa_1||A^{1/2}\bar{v}||^2 + \mu\eta^*\kappa_2||A_0^{1/2}\bar{s}||^2 = \frac{\mu}{2(\tau^*)^2} \text{Re}\bar{\lambda}. \tag{38}
\]

Now, multiplying (32) by $2i\beta\bar{u}$ and (33) by $\bar{\lambda}\bar{u}$ and subtracting resultants from each other, we obtain

\[
||A^{1/2}\bar{u}||^2 - (\eta^*)^2||A^{1/2}\bar{v}||^2 = 0 \quad \text{and} \quad ||\bar{u}||^2 - (\eta^*)^2||\bar{v}||^2 = 0.
\]

Thus (38) implies that

\[
\frac{\mu}{2(\tau^*)^2} \text{Re}\bar{\lambda} = (4 - \frac{\kappa_1}{\eta^*})||A^{1/2}\bar{u}||^2 + \mu\kappa_2\eta^*||A_0^{1/2}\bar{s}||^2
\]

which is positive since $4 - \frac{\kappa_1}{\eta^*} > 0$. We have $\text{Re}\lambda(\tau^*) > 0$ for $\beta > 0$, and thus, by the Hopf-bifurcation theorem in [1, 4], there exists a family of periodic solutions which bifurcates from the stationary solution as $\tau$ passes $\tau^*$.

We shall now show that there exists a unique $\tau^* > 0$ such that $(0, 0, 0, 0, \eta^*, \tau^*)$ is a Hopf point; thus $\tau^*$ is the origin of a branch of nontrivial periodic orbits.
Lemma 4.4. Suppose that $4 - \frac{a_1}{q} > 0$. Let $G_{\beta}$ and $\hat{G}_{\beta}$ be Green functions of the differential operator $A + i\beta$ satisfying (16) and (20), respectively. Then, the expression $4\Re(G_{\beta}(\eta^*, \eta^*)) - \frac{a_1}{q}2\Re(\hat{G}_{\beta}(\eta^*, \eta^*))$ and $\Re(J_{\beta}(\eta^*, \eta^*))$ are strictly decreasing in $\beta \in \mathbb{R}^+$ with

$$\Re G_0(\eta^*, \eta^*) = G(\eta^*, \eta^*), \quad \lim_{\beta \to \infty} \Re G_\beta(\eta^*, \eta^*) = 0.$$ 

Moreover, $-4\Im G_{\beta}(\eta^*, \eta^*) + \frac{a_1}{q}2 \Im(\hat{G}_{\beta}(\eta^*, \eta^*)) - \frac{a_2}{q}2 \Im(\hat{J}_{\beta}(\eta^*, \eta^*)) > 0$ for $\beta > 0$

The proof is in [3].

Theorem 4.5. Under the same condition as in Theorem 4.2, for a unique critical point $\tau^* > 0$, there exists a unique, purely imaginary eigenvalue $\lambda = i\beta$ of (11) with $\beta > 0$.

Proof: We only need to show that the function $(u, v, w, s, \beta, \tau) \mapsto E(u, v, w, s, i\beta, \tau)$ has a unique zero with $\beta > 0$ and $\tau > 0$. This means solving the system of equations (11) with $\lambda = i\beta$, $u = a - \mu G(\cdot, \eta^*)$, $v = p + \frac{\mu}{q} G(\cdot, \eta^*)$, $w = b - J(\cdot, \eta^*)$ and $s = q + \frac{1}{q} \hat{J}(\cdot, \eta^*)$,

$$\begin{cases}
(A + i\beta)a = -\mu \delta_{\eta^*}, \\
(A + i\beta)p = \frac{\mu}{q} \delta_{\eta^*}, \\
(A_0 + i\beta)b = -\delta_{\eta^*}, \\
(A_0 + i\beta)q = \frac{1}{q} \delta_{\eta^*}, \\
\frac{i\beta}{q} = 4(a(\eta^*) + \mu G(\eta^*, \eta^*) + \gamma'(\eta^*)) + \kappa_1(p(\eta^*) - \frac{\mu}{q} \hat{G}(\eta^*, \eta^*) + \gamma'(\eta^*)) \\
-\kappa_2(q(\eta^*) - \frac{1}{q} \hat{J}(\eta^*, \eta^*) + \gamma'(\eta^*)). 
\end{cases}$$

The real and imaginary parts of the above equation are given by

$$\begin{cases}
\frac{\beta}{q} = 4\Im(-\mu G_{\beta}(\eta^*, \eta^*)) + \frac{\mu a_1}{q}2 \Im(\hat{G}_{\beta}(\eta^*, \eta^*)) - \frac{a_2}{q}2 \Im(\hat{J}_{\beta}(\eta^*, \eta^*)) \\
0 = 4(\Re(-\mu G_{\beta}(\eta^*, \eta^*)) + \mu G(\eta^*, \eta^*) + \gamma'(\eta^*)) \\
+ \kappa_1(\Re(\frac{\mu}{q} \hat{G}_{\beta}(\eta^*, \eta^*)) - \frac{\mu}{q} \hat{G}(\eta^*, \eta^*) + \gamma'(\eta^*)) \\
- \kappa_2(\Re(\frac{1}{q} \hat{J}_{\beta}(\eta^*, \eta^*)) - \frac{1}{q} \hat{J}(\eta^*, \eta^*) + \gamma'(\eta^*)). 
\end{cases}$$

Since $4\Im(-\mu G_{\beta}(\eta^*, \eta^*)) + \kappa_1(\frac{\mu}{q} \Im(\hat{G}_{\beta}(\eta^*, \eta^*)) - \frac{a_2}{q}2 \Im(\hat{J}_{\beta}(\eta^*, \eta^*)) > 0$ if $4 - \frac{a_1}{q} > 0$ by Lemma 4.4, there is a critical point $\tau^*$, provided the existence of $\beta$. We now define

$$T(\beta) = 4(\Re(-\mu G_{\beta}(\eta^*, \eta^*)) + \mu G(\eta^*, \eta^*) + \gamma'(\eta^*)) \\
+ \kappa_1(\Re(\frac{\mu}{q} \hat{G}_{\beta}(\eta^*, \eta^*)) - \frac{\mu}{q} \hat{G}(\eta^*, \eta^*) + \gamma'(\eta^*)) \\
- \kappa_2(\Re(\frac{1}{q} \hat{J}_{\beta}(\eta^*, \eta^*)) - \frac{1}{q} \hat{J}(\eta^*, \eta^*) + \gamma'(\eta^*)).$$
If $4 - \frac{\alpha_1}{\eta} > 0$, $T(\beta)$ is a decreasing function of $\beta$ by Lemma 4.4 and $T(0) = 4\gamma'(\eta^*) + \kappa_1 \gamma'(\eta^*) - \kappa_2 \hat{\alpha}'(\eta^*) < 0$ if $\mu \kappa_1 < \kappa_2$. Moreover,

$$\lim_{\beta \to \infty} T(\beta) = (4 - \frac{\alpha_1}{\eta}) \mu \hat{G}(\eta^*, \eta^*) + \kappa_1 \gamma'(\eta^*) + \kappa_2 \left( \frac{J(\eta^*, \eta^*)}{\eta^*} - \hat{\alpha}'(\eta^*) \right).$$

We have

$$\frac{J(\eta^*, \eta^*)}{\eta^*} - \hat{\alpha}'(\eta^*) = \frac{2 \sinh(\frac{1-\eta^*}{2})}{\sinh 1} \left( \frac{1}{\eta^*} \sinh \eta^* \cosh(\frac{1-\eta^*}{2}) - \sinh(\frac{1-3\eta^*}{2}) \right)$$

$$> \frac{2}{\sinh 1} \sinh(\frac{1-\eta^*}{2}) \left( \cosh(\frac{1-\eta^*}{2})(1 + \sinh \eta^* - \sinh(\frac{1-\eta^*}{2}) \cosh \eta^*) \right)$$

$$> \frac{2}{\sinh 1} (\sinh(\frac{1-\eta^*}{2}))^2 (1 + \sinh \eta^* - \cosh \eta^*)$$

is positive for $0 < \eta^* < \frac{1}{3}$ since $\frac{\sinh \eta^*}{\eta^*} > 1$ and $\cosh \eta^* > \sinh \eta^*$. Therefore $\lim_{\beta \to \infty} T(\beta)$ is positive if $4 > \frac{\alpha_1}{\eta}$. Hence, there exists a unique $\beta > 0$ and $\tau^*$ for $0 < \eta^* < \frac{1}{3}$.

The following theorem summarizes the results above.

**Theorem 4.6.** Suppose that $\frac{\mu \cosh \frac{\sqrt{1+\mu} - \sqrt{1+\mu}}{1+\mu}}{\sinh \sqrt{1+\mu}} < \frac{1}{2} - \alpha_0 < \frac{\mu}{1+\mu}$ and $\mu \kappa_1 < \kappa_2$. Then (8) and (6) have at least one stationary solution $(u^*, v^*, w^*, s^*, \eta^*)$ where $u^* = v^* = w^* = s^* = 0$, $0 < \eta^* < \frac{1}{3}$ and $(a^*, p^*, b^*, q^*, \eta^*)$ for all $\tau$, respectively. Moreover, assume that $4 > \frac{\alpha_1}{\eta}$. Then there exists a unique $\tau^*$ such that the linearization $-\tilde{A} + \tau^* \tilde{B}$ has a purely imaginary pair of eigenvalues. The point $(0, 0, 0, 0, \eta^*, \tau^*)$ is then a Hopf point for (8), and there exists a $C^0$-curve of nontrivial periodic orbits for (8) and (6), bifurcating from $(0, 0, 0, 0, \eta^*, \tau^*)$ and $(u^*, v^*, w^*, s^*, \eta^*, \tau^*)$, respectively.

**REFERENCES**


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