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# Three-Dimensional Theory of Nonlinear-Elastic Bodies Stability under Finite Deformations

**Yu. I. Dimitrienko**

Computational Mathematics and Mathematical Physics Department
Bauman Moscow State Technical University, 2 Baumanskaya
Street, 5, 105005 Moscow, Russia



**Abstract**

The computation problem of elastic structures stability is one of main problems of solid mechanics. Traditional methods of stability calculation are based on applying the theory of two-dimensional shell structures, in general, the classical Kirchhoff-Love theory. The developed methods for solving three-dimensional problems of the stability theory allow us to expand the frames of solved stability problems and to increase the accuracy of obtained solutions. The purpose of the paper is to derive generalized three-dimensional equations of the stability theory of nonlinearly elastic bodies with finite deformations for a wide class of nonlinear elastic models. For this, the method of a varied configuration and the universal method of representation of nonlinearly elastic continua models on the base of energetic couples of stress and strain tensors were applied. It is shown that for two of the tensor couples the stability theory relations give an explicit analytical expression without calculation of eigenvalues of the stretch tensor.

**Keywords:** three-dimensional theory of elastic stability, energetic stress and strain tensors, finite elastic deformations

Equations of the shell stability theory with small deformations for different cases are usually derived with the help of a number of hypotheses and assumptions [1, 2, 6, 7], because the stability equations even for continua with small deformations follow from general nonlinear equations of the elasticity theory with finite deformations, which are complicated enough in the general statement and not definitive. Due to development of powerful computers using finite-element methods, there appears an interest in three-dimensional problems of the stability theory. To derive generalized three-dimensional equations of the theory for nonlinear-elastic solids with finite deformations, we apply the advanced method of a varied configuration and the universal method of representation of solid models on the base of energetic couples of stress and strain tensors [3, 4].

## The varied configuration

Let us consider the general case of finite deformations of elastic solids [3-5]. Together with actual configuration $K$ of a solid continuum at time $t$, we introduce one more actual configuration $\widehat{K}$, which is called varied and differs from the true configuration $K$ by a small displacement. The configuration $\widehat{K}$ is used for search of possible not unique solution, the existence of which means that there appears an instability of the body. Radius-vector $\hat{\mathbf{x}}$ of a point $M$ in $\widehat{K}$ is connected with $\mathbf{x}$ of the same point in $K$ by $\hat{\mathbf{x}} = \mathbf{x} + \delta\mathbf{x}$, where $\delta\mathbf{x}$ is the radius-vector variation determined in the following way. Let $f(\xi)$ be a smooth scalar function defined within interval $0 \le \xi \le \xi_m$, then in a small neighborhood of point $\xi = 0$, this function can be considered as the linear dependence: $f(\xi) = f(0) + \xi f_\xi(0)$, where $f_\xi(0) = \frac{df}{d\xi}(0) = \lim\limits_{\xi\to 0}\frac{f(\xi) - f(0)}{\xi}$ is the derivative of the function in zero. With use of this representation, let us consider the radius-vector $\hat{\mathbf{x}}$ as a function not only of Lagrangian coordinates $X^i$ of the material point and time $t$, but also of the additional parameter $\xi$ (fictitious time), and this function is assumed to be linear: $\hat{\mathbf{x}} \equiv \mathbf{x}(X^i,t,\xi) = \mathbf{x}(X^i,t,0) + \xi\mathbf{w}(X^i,t)$, $\mathbf{x}(X^i,t,0) = \mathbf{x}(X^i,t)$, $\mathbf{w} \equiv (d/d\xi)\mathbf{x}(X^i,t,\xi)\big|_{\xi=0}$. Hence, we find the radius-vector variation $\delta\mathbf{x}$ as a linear function of $\xi$: $\delta\mathbf{x} = \xi\mathbf{w}$. The body location in the actual configuration $K$ is assumed to be known (i.e. $\mathbf{x}$ is known), then the stability theory problem consists in finding the varied configuration $\widehat{K}$, i.e. $\mathbf{w}$ (or $\delta\mathbf{x}$).

## Kinematics of a varied configuration

On differentiating the relations with respect to $X^i$, we obtain the local basis vectors in configuration $\widehat{K}$:

$$\hat{\mathbf{r}}_i = \frac{\partial\hat{\mathbf{x}}}{\partial X^i} = \frac{\partial\mathbf{x}}{\partial X^i} + \xi\frac{\partial\mathbf{w}}{\partial X^i} = \mathbf{r}_i + \xi\frac{\partial\mathbf{w}}{\partial X^i} = \mathbf{r}_i\cdot(\mathbf{E} + \xi\nabla\otimes\mathbf{w}). \qquad (1)$$

For $\hat{\mathbf{r}}_i$ at point $\xi = 0$, we can also use the linear representation $\hat{\mathbf{r}}_i = \mathbf{r}_i + \xi\mathbf{r}_{i\xi}$, $\mathbf{r}_{i\xi} \equiv \equiv \frac{d}{d\xi}\mathbf{r}_i\big|_{\xi=0}$. On comparing these relations with (1), we get $\mathbf{r}_{i\xi} = \mathbf{r}_i\cdot\nabla\otimes\mathbf{w}$. The reciprocal basis vectors $\hat{\mathbf{r}}^i$ have the form $\hat{\mathbf{r}}^i = \mathbf{r}^i - \xi\mathbf{r}^i\cdot\nabla\otimes\mathbf{w}^{\mathrm{T}}$. The expression for the vectors in the neighborhood of point $\xi = 0$ has the form

$$\hat{\mathbf{r}}^i = \mathbf{r}^i + \xi \mathbf{r}^i_\xi, \quad \mathbf{r}^i_\xi \equiv \frac{d}{d\xi} \hat{\mathbf{r}}^i \Big|_{\xi=0} . \tag{2}$$

Comparing these formulae, we obtain the expression $\mathbf{r}^i_\xi = -\mathbf{r}^i \cdot \nabla \otimes \mathbf{w}^{\mathrm{T}}$. These relations allow us to represent variations of local basis vectors $\delta \mathbf{r}_i$ and $\delta \mathbf{r}^i$ in terms of gradient $\nabla \otimes \mathbf{w}$: $\hat{\mathbf{r}}_i = \mathbf{r}_i + \delta \mathbf{r}_i$, $\hat{\mathbf{r}}^i = \mathbf{r}^i + \delta \mathbf{r}^i$, $\delta \mathbf{r}_i = \xi \mathbf{r}_{i\xi}$, $\delta \mathbf{r}^i = \xi \mathbf{r}^i_\xi$. All formulae for variations of stress and strain tensors have the similar form. Therefore, for finding these variations it is sufficient to find only derivatives with respect to $\xi$, which are called convective derivatives, because they determine the change of values at point $M$ at passage from configuration $K$ into $\widehat{K}$.

## Convective derivative of the strain gradient

The strain gradient $\widehat{\mathbf{F}}$ in configur-ation $\widehat{K}$ is defined similarly to the strain gradient in $K$: $\widehat{\mathbf{F}} = \hat{\mathbf{r}}_i \otimes \overset{0}{\mathbf{r}}{}^i$, where $\overset{0}{\mathbf{r}}{}^i$ are local vectors of the reciprocal basis in the reference configuration. With account of (2) we then obtain its linear representation in the neighborhood of point $\xi = 0$:

$$\widehat{\mathbf{F}} = \mathbf{r}_i \otimes \overset{0}{\mathbf{r}}{}^i + \xi \nabla \otimes \mathbf{w}^{\mathrm{T}} \cdot \mathbf{r}_i \otimes \overset{0}{\mathbf{r}}{}^i = \mathbf{F} + \xi \nabla \otimes \mathbf{w}^{\mathrm{T}} \cdot \mathbf{F}, \qquad \mathbf{F}_\xi = \frac{d}{d\xi} \widehat{\mathbf{F}} \Big|_{\xi=0} = \overset{0}{\nabla} \otimes \mathbf{w}^{\mathrm{T}} . \tag{3}$$

The inverse gradient $\widehat{\mathbf{F}}^{-1}$ in $\widehat{K}$ is determined by the formula: $\widehat{\mathbf{F}}^{-1} = \overset{0}{\mathbf{r}}_i \otimes \hat{\mathbf{r}}^i =$ $= \mathbf{F}^{-1} - \mathbf{F}^{-1} \cdot \nabla \otimes \mathbf{w}^{\mathrm{T}}$, $\mathbf{F}^{-1}_\xi = \frac{d}{d\xi} \widehat{\mathbf{F}}^{-1} \Big|_{\xi=0} = -\mathbf{F}^{-1} \cdot \nabla \otimes \mathbf{w}^{\mathrm{T}}$. The derivative of smooth scalar functions $\Phi_1(\widehat{\mathbf{F}})$ and $\Phi_2(\widehat{\mathbf{F}})$ with respect to $\xi$ has the form

$$\Phi_1(\mathbf{F})_\xi = \frac{\partial \Phi_1}{\partial \widehat{\mathbf{F}}} \Big|_{\xi=0} \cdot\cdot \mathbf{F}^{\mathrm{T}}_\xi, \quad (\Phi_1 \Phi_2)_\xi = \Phi_{1\xi} \widehat{\Phi}_2 \Big|_{\xi=0} + \widehat{\Phi}_1 \Big|_{\xi=0} \Phi_{2\xi} = \Phi_{1\xi} \Phi_2 + \Phi_1 \Phi_{2\xi} . \tag{4}$$

In particular, choosing $\Phi_1 = \sqrt{g / \overset{0}{g}}$, where $g = \det(g_{ij}), \overset{0}{g} = \det(\overset{0}{g}_{ij})$ are determinants of metric matrices: $\overset{0}{g}_{ij} = \overset{0}{\mathbf{r}}_i \cdot \overset{0}{\mathbf{r}}_j$, $g_{ij} = \mathbf{r}_i \cdot \mathbf{r}_j$, with account of the continuity equation in Lagrangian description [4] ( $\sqrt{g / \overset{0}{g}} = \det \widehat{\mathbf{F}}$ ) and $\widehat{\mathbf{F}} \Big|_{\xi=0} = \mathbf{F}$, we obtain

$$(\sqrt{g / \overset{0}{g}})_\xi = \frac{\partial}{\partial \widehat{\mathbf{F}}} (\sqrt{g / \overset{0}{g}})_{\xi=0} \cdot\cdot \mathbf{F}^{\mathrm{T}}_\xi = (\det \widehat{\mathbf{F}})_{\xi=0} \widehat{\mathbf{F}}^{-1\mathrm{T}} \Big|_{\xi=0} \cdot \mathbf{F}^{\mathrm{T}} \cdot\cdot \nabla \otimes \mathbf{w} = \sqrt{g / \overset{0}{g}} \nabla \cdot \mathbf{w} . \tag{5}$$

Formulae (4) and (5) allow us to determine the convective derivative without use of the varied configuration $\widehat{K}$, but immediately by formal rules of differentiation of tensors with respect to the fictitious time chosen as parameter $\xi$ [4].

## Convective derivatives of eigenvectors and eigenvalues of the stretch tensors

Let us determine convective derivatives $\lambda_{\alpha\xi}, \overset{0}{\mathbf{p}}_{\alpha\xi}, \mathbf{p}_{\alpha\xi}$ of eigenvalues $\lambda_\alpha$ and eigenvectors of the right stretch tensor $\mathbf{U} = \mathbf{F}^{\mathrm{T}} \cdot \mathbf{F}$, and derivatives $\mathbf{U}_\xi$ and $\mathbf{O}_\xi$. We use the properties of eigenvalues and eigenvectors [3]: $\mathbf{U} = \sum_{\alpha=1}^{3} \lambda_\alpha \overset{0}{\mathbf{p}}_\alpha \otimes \overset{0}{\mathbf{p}}_\alpha$,

$$\mathbf{U}^2 = \sum_{\alpha=1}^{3} \lambda_\alpha^2 \overset{0}{\mathbf{p}}_\alpha \otimes \overset{0}{\mathbf{p}}_\alpha, \quad \overset{0}{\mathbf{p}}_\alpha \cdot \overset{0}{\mathbf{p}}_\beta = \delta_{\alpha\beta}, \quad \mathbf{U}^2 \cdot \overset{0}{\mathbf{p}}_\alpha = \lambda_\alpha^2 \overset{0}{\mathbf{p}}_\alpha, \quad \mathbf{U}^2 = \mathbf{F}^{\mathrm{T}} \cdot \mathbf{F}. \tag{6}$$

Differentiation of the fourth formula in (6) with respect to $\xi$ gives $\mathbf{U}_\xi^2 = = 2\mathbf{F}^{\mathrm{T}} \cdot \boldsymbol{\varepsilon}(\mathbf{w}) \cdot \mathbf{F}$. Differentiating the second formula in (6) with respect to $\xi$ and assuming $\beta = \alpha$ there, we find $\overset{0}{\mathbf{p}}_\alpha \cdot \overset{0}{\mathbf{p}}_{\alpha\xi} = 0$, i.e. vectors $\overset{0}{\mathbf{p}}_\alpha$ and $\overset{0}{\mathbf{p}}_{\alpha\xi}$ are orthogonal. Differentiating the third formula in (6) with respect to $\xi$, we have

$$\mathbf{U}_\xi^2 \cdot \overset{0}{\mathbf{p}}_\alpha + \mathbf{U}^2 \cdot \overset{0}{\mathbf{p}}_{\alpha\xi} = \lambda_{\alpha\xi}^2 \overset{0}{\mathbf{p}}_\alpha + \lambda_\alpha^2 \overset{0}{\mathbf{p}}_{\alpha\xi}. \tag{7}$$

On multiplying this equation by $\overset{0}{\mathbf{p}}_\alpha$, we obtain $\overset{0}{\mathbf{p}}_\alpha \cdot \mathbf{U}_\xi^2 \cdot \overset{0}{\mathbf{p}}_\alpha + \overset{0}{\mathbf{p}}_\alpha \cdot \mathbf{U}^2 \cdot \overset{0}{\mathbf{p}}_{\alpha\xi} = = \lambda_{\alpha\xi}^2 \overset{0}{\mathbf{p}}_\alpha \cdot \overset{0}{\mathbf{p}}_\alpha + \lambda_\alpha^2 \overset{0}{\mathbf{p}}_\alpha \cdot \overset{0}{\mathbf{p}}_{\alpha\xi}$. With account of (6) the second summands at the right and left sides of the equation vanish, thus $\lambda_{\alpha\xi}^2 = \overset{0}{\mathbf{p}}_\alpha \cdot \mathbf{U}_\xi^2 \cdot \overset{0}{\mathbf{p}}_\alpha$. Hence, with account of (4) we find $\lambda_{\alpha\xi} = (1/\lambda_\alpha) \overset{0}{\mathbf{p}}_\alpha \cdot \mathbf{F}^{\mathrm{T}} \cdot \boldsymbol{\varepsilon}(\mathbf{w}) \cdot \mathbf{F} \cdot \overset{0}{\mathbf{p}}_\alpha$. Using the polar decomposition $\mathbf{F} = \mathbf{O} \cdot \mathbf{U}$ and $\mathbf{p}_\alpha = \mathbf{O} \cdot \overset{0}{\mathbf{p}}_\alpha$ [4], we get $\mathbf{F} \cdot \overset{0}{\mathbf{p}}_\alpha = \mathbf{O} \cdot \mathbf{U} \cdot \overset{0}{\mathbf{p}}_\alpha = \lambda_\alpha \mathbf{p}_\alpha$. Then we find the final formulae for the convective derivatives of eigenvalues: $\lambda_{\alpha\xi} = \lambda_\alpha \mathbf{p}_\alpha \cdot \boldsymbol{\varepsilon}(\mathbf{w}) \cdot \mathbf{p}_\alpha$ and $\lambda_{\alpha\xi}^2 = 2\lambda_\alpha^2 \mathbf{p}_\alpha \cdot \boldsymbol{\varepsilon}(\mathbf{w}) \cdot \mathbf{p}_\alpha$. Multiplying the relation (7) by $\overset{0}{\mathbf{p}}_\beta$:

$$\overset{0}{\mathbf{p}}_\beta \cdot \mathbf{U}_\xi^2 \cdot \overset{0}{\mathbf{p}}_\alpha + \overset{0}{\mathbf{p}}_\beta \cdot \mathbf{U}^2 \cdot \overset{0}{\mathbf{p}}_{\alpha\xi} = \lambda_{\alpha\xi}^2 \overset{0}{\mathbf{p}}_\beta \cdot \overset{0}{\mathbf{p}}_\alpha + \lambda_\alpha^2 \overset{0}{\mathbf{p}}_\beta \cdot \overset{0}{\mathbf{p}}_{\alpha\xi}, \ \alpha \neq \beta, \tag{8}$$

we obtain (all $\lambda_\alpha$ are assumed to be different): $\overset{0}{\mathbf{p}}_\beta \cdot \overset{0}{\mathbf{p}}_{\alpha\xi} = \overset{0}{\mathbf{p}}_\beta \cdot \mathbf{U}_\xi^2 \cdot \overset{0}{\mathbf{p}}_\alpha / (\lambda_\alpha^2 - \lambda_\beta^2)$, $\alpha \neq \beta$. Resolving the vectors $\overset{0}{\mathbf{p}}_{\alpha\xi}$ for basis $\overset{0}{\mathbf{p}}_\alpha$, we have

$$\overset{0}{\mathbf{p}}_{\alpha\xi} = \sum_{\beta=1}^{3} (\overset{0}{\mathbf{p}}_{\alpha\xi} \cdot \overset{0}{\mathbf{p}}_\beta) \overset{0}{\mathbf{p}}_\beta = \sum_{\alpha\neq\beta,\beta=1}^{3} \frac{\overset{0}{\mathbf{p}}_\beta \cdot \mathbf{U}_\xi^2 \cdot \overset{0}{\mathbf{p}}_\alpha}{\lambda_\alpha^2 - \lambda_\beta^2} \overset{0}{\mathbf{p}}_\beta, \tag{9}$$

The final formula for the derivatives of eigenvectors has the form

$$\overset{0}{\mathbf{p}}_{\alpha\xi} = 2 \sum_{\alpha\neq\beta,\beta=1}^{3} \lambda_\alpha \lambda_\beta \frac{\mathbf{p}_\beta \cdot \boldsymbol{\varepsilon}(\mathbf{w}) \cdot \mathbf{p}_\alpha}{\lambda_\alpha^2 - \lambda_\beta^2} \overset{0}{\mathbf{p}}_\beta. \tag{10}$$

Using the resolution of tensor $\mathbf{V}$ for the eigenbasis: $\mathbf{V}^2=\sum_{\alpha=1}^{3}\lambda_\alpha^2\mathbf{p}_\alpha\otimes\mathbf{p}_\alpha$, $\mathbf{V}^2\cdot\mathbf{p}_\alpha=\lambda_\alpha^2\mathbf{p}_\alpha$, we obtain $\mathbf{p}_\beta\cdot\mathbf{V}_\xi^2\cdot\mathbf{p}_\alpha+\mathbf{p}_\beta\cdot\mathbf{V}^2\cdot\mathbf{p}_{\alpha\xi}=\mathbf{p}_\beta\cdot\mathbf{V}_\xi^2\cdot\mathbf{p}_\alpha+\mathbf{p}_\beta\cdot\mathbf{V}^2\cdot\mathbf{p}_{\alpha\xi}=$ $=\lambda_\alpha^2\mathbf{p}_\beta\cdot\mathbf{p}_{\alpha\xi}$, being the analog of formula (8). Hence, we have

$$\mathbf{p}_\beta\cdot\mathbf{p}_{\alpha\xi}=\frac{\mathbf{p}_\beta\cdot V_\xi^2\cdot\mathbf{p}_\alpha}{\lambda_\alpha^2-\lambda_\beta^2},\qquad \mathbf{p}_{\alpha\xi}=\sum_{\beta=1}^{3}(\mathbf{p}_{\alpha\xi}\cdot\mathbf{p}_\beta)\mathbf{p}_\beta=2\sum_{\alpha\neq\beta\beta=1}^{3}\frac{\mathbf{p}_\beta\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{w})\cdot\mathbf{p}_\alpha}{\lambda_\alpha^2-\lambda_\beta^2}\mathbf{p}_\beta. \quad (11)$$

Using formulae for $\lambda_{\alpha\xi}$, $\overset{0}{\mathbf{p}}_{\alpha\xi}$ and $\mathbf{p}_{\alpha\xi}$, we find convective derivatives of stretch tensors $\mathbf{U}_\xi^n$ and $\mathbf{V}_\xi^n$ to the *n*th power, and the derivative of the rotation tensor $\mathbf{O}_\xi$ with differentiating the formulae of resolution of the tensors for the eigenbasis:

$$\mathbf{U}_\xi^n=\sum_{\alpha=1}^{3}(n\lambda_\alpha^{n-1}\lambda_{\alpha\xi}\overset{0}{\mathbf{p}}_\alpha\otimes\overset{0}{\mathbf{p}}_\alpha+\lambda_\alpha^n(\overset{0}{\mathbf{p}}_{\alpha\xi}\otimes\overset{0}{\mathbf{p}}_\alpha+\overset{0}{\mathbf{p}}_\alpha\otimes\overset{0}{\mathbf{p}}_{\alpha\xi}))=\sum_{\alpha,\beta=1}^{3}\overset{(n)}{\mathrm{U}}_{\alpha\beta}(\mathbf{p}_\alpha\cdot\boldsymbol{\varepsilon}\cdot\mathbf{p}_\beta)\overset{0}{\mathbf{p}}_\alpha\otimes\overset{0}{\mathbf{p}}_\beta,$$

$$\mathbf{V}_\xi^n=\sum_{\alpha,\beta=1}^{3}(\overset{(n)}{\mathrm{V}}_{\alpha\beta}(\mathbf{p}_\alpha\cdot\boldsymbol{\varepsilon}(\mathbf{w})\cdot\mathbf{p}_\beta)+\overset{(n)}{\tilde{\mathrm{V}}}_{\alpha\beta}(\mathbf{p}_\alpha\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{w})\cdot\mathbf{p}_\beta))\mathbf{p}_\alpha\otimes\mathbf{p}_\beta,\quad \overset{(n)}{\mathrm{V}}_{\alpha\beta}=\lambda_\alpha^{\mathrm{n}}\mathrm{n}\delta_{\alpha\beta},\quad (12)$$

$$\mathbf{O}_\xi=\sum_{\alpha=1}^{3}O_{\alpha\beta}\mathbf{p}_\alpha\cdot\overset{0}{\mathbf{p}}_\beta,\qquad O_{\alpha\beta}=2(\lambda_\alpha\lambda_\beta(\mathbf{p}_\alpha\cdot\boldsymbol{\varepsilon}\cdot\mathbf{p}_\beta)+\mathbf{p}_\beta\cdot\tilde{\boldsymbol{\varepsilon}}\cdot\mathbf{p}_\alpha)\frac{1-\delta_{\alpha\beta}}{\lambda_\alpha^2-\lambda_\beta^2}$$

$$\overset{(n)}{\tilde{\mathrm{V}}}_{\alpha\beta}=\frac{2(1-\delta_{\alpha\beta})}{\lambda_\alpha{}^2-\lambda_\beta{}^2}\left(\lambda_\beta^{\mathrm{n}}-\lambda_\alpha^{\mathrm{n}}\right),\ \overset{(n)}{\mathrm{U}}_{\alpha\beta}=\lambda_\alpha^{\mathrm{n}}\mathrm{n}\delta_{\alpha\beta}+\frac{2(1-\delta_{\alpha\beta})\lambda_\alpha\lambda_\beta}{\lambda_\alpha{}^2-\lambda_\beta{}^2}\left(\lambda_\beta^{\mathrm{n}}-\lambda_\alpha^{\mathrm{n}}\right).$$

## Convective derivatives of energetic and quasienergetic strain tensors

Using formulae (12) and generalized representations [4] for energetic and quasienergetic strain tensors $\overset{(n)}{\mathbf{C}}$ and $\overset{(n)}{\mathbf{A}}$, we find the expressions for their convective derivatives

$$\overset{(n)}{\mathbf{C}}_\xi={}^4\overset{(n)}{\mathbf{U}}\cdot\cdot\boldsymbol{\varepsilon}(\mathbf{w}),\ \overset{(n)}{\mathbf{A}}_\xi=\frac{1}{\mathrm{n}-\mathrm{III}}\mathbf{V}_\xi^n={}^4\overset{(n)}{\mathbf{V}}\cdot\cdot\boldsymbol{\varepsilon}(\mathbf{w})+{}^4\overset{(n)}{\tilde{\mathbf{V}}}\cdot\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{w}), \quad (13)$$

$${}^4\overset{(n)}{\mathbf{U}}=\frac{1}{\mathrm{n}-\mathrm{III}}\sum_{\alpha,\beta=1}^{3}\overset{(n-III)}{\mathrm{U}}_{\alpha\beta}\overset{0}{\mathbf{p}}_\alpha\otimes\overset{0}{\mathbf{p}}_\beta\otimes\mathbf{p}_\alpha\otimes\mathbf{p}_\beta,\ {}^4\overset{(n)}{\mathbf{V}}=\frac{1}{\mathrm{n}-\mathrm{III}}\sum_{\alpha,\beta=1}^{3}\overset{(n-III)}{\mathrm{V}}_{\alpha\beta}\mathbf{p}_\alpha\otimes\mathbf{p}_\beta\otimes\mathbf{p}_\alpha\otimes\mathbf{p}_\beta.$$

## The stress tensors in the varied configuration

Consider models $A_n$ of a nonlinear, in general, anisotropic continuum, which are determined by constitutive relations [4]: $\overset{(n)}{\mathbf{T}}=\sum_{\gamma=1}^{\mathrm{r}}\varphi_\gamma\mathrm{I}_{\gamma\mathbf{C}}^{(s)},\quad \varphi_\gamma=\partial\psi/\partial I_\gamma^{(s)}$, $I_{\gamma\mathbf{C}}^{(s)}=\partial I_\gamma^{(s)}/\partial\overset{(n)}{\mathbf{C}},\ \mathrm{I}_\gamma^{(s)}=\mathrm{I}_\gamma^{(s)}(\overset{(n)}{\mathbf{C}})$. Then with account of (5) we have

$$\overset{(n)}{\mathbf{T}}_\xi = {}^4\overset{(n)}{\mathbf{H}}{}^{(s)} \cdot\cdot \overset{(n)}{\mathbf{C}}_\xi = {}^4\overset{(n)}{\mathbf{H}}{}^{(s)} \cdot\cdot {}^4\overset{(n)}{\mathbf{U}} \cdot\cdot \boldsymbol{\varepsilon}(\mathbf{w}),$$

$${}^4\overset{(n)}{\mathbf{H}}{}^{(s)} = \sum_{\gamma,\beta=1}^{r} \frac{\partial^2 \psi}{\partial I_\gamma^{(s)} \partial I_\beta^{(s)}} I_{\gamma\mathbf{C}}^{(s)} \otimes I_{\beta\mathbf{C}}^{(s)} + \sum_{\gamma=1}^{r} \varphi_\gamma \frac{\partial^2 I_\gamma^{(s)}}{\partial \overset{(n)}{\mathbf{C}} \partial \overset{(n)}{\mathbf{C}}}. \quad (14)$$

To calculate the derivatives $\mathbf{T}_\xi$ and $\mathbf{P}_\xi$ of the Cauchy and Piola-Kirchhoff stress tensors we use the relations between tensors $\overset{(n)}{\mathbf{T}}$ and $\mathbf{T}$, $\mathbf{P}$ [4]:

$$\mathbf{T} = {}^4\overset{(n)}{\mathbf{E}} \cdot\cdot \overset{(n)}{\mathbf{T}}, \qquad \mathbf{P} = {}^4\overset{(n)}{\mathbf{E}}{}^{0} \cdot\cdot \overset{(n)}{\mathbf{T}}, \qquad {}^4\overset{(n)}{\mathbf{E}} = \sum_{\alpha,\beta=1}^{3} \overset{(n)}{\mathrm{E}}_{\alpha\beta} \mathbf{p}_\alpha \otimes \mathbf{p}_\beta \otimes \overset{0}{\mathbf{p}}_\beta \otimes \overset{0}{\mathbf{p}}_\alpha,$$

$${}^4\overset{(n)}{\mathbf{E}}{}^{0} = (\overset{0}{\rho}/\rho)\mathbf{F}^{-1} \cdot {}^4\overset{(n)}{\mathbf{E}} = \sum_{\alpha,\beta=1}^{3} \overset{(n)}{\mathrm{E}}{}^{0}_{\alpha\beta} \mathbf{p}_\alpha \otimes \mathbf{p}_\beta \otimes \overset{0}{\mathbf{p}}_\beta \otimes \overset{0}{\mathbf{p}}_\alpha, \quad \overset{(n)}{\mathrm{E}}{}^{0}_{\alpha\beta} = \sqrt{\overset{0}{\mathrm{g}}/\mathrm{g}}\, \overset{(n)}{\mathrm{E}}_{\alpha\beta} / \lambda_\alpha. \quad (15)$$

Components of energetic equivalence tensors ${}^4\overset{(n)}{\mathbf{E}}$ and ${}^4\overset{(n)}{\mathbf{E}}{}^{0}$ depend only on $\lambda_\alpha$ and $\lambda_\beta$ [4]. Differentiating the formulae (12) with respect to $\xi$, we get

$$\mathbf{T}_\xi = {}^4\overset{(n)}{\mathbf{E}}_\xi \cdot\cdot \overset{(n)}{\mathbf{T}} + {}^4\overset{(n)}{\mathbf{E}} \cdot\cdot \overset{(n)}{\mathbf{T}}_\xi, \quad \mathbf{P}_\xi = {}^4\overset{(n)}{\mathbf{E}}{}^{0}_\xi \cdot\cdot \overset{(n)}{\mathbf{T}} + {}^4\overset{(n)}{\mathbf{E}}{}^{0} \cdot\cdot \overset{(n)}{\mathbf{T}}_\xi. \quad (16)$$

## Formulation of the stability problem for a nonlinear-elastic body

Let us write the equilibrium equation in reference configuration $\overset{0}{K}$: $\overset{0}{\nabla} \cdot \mathbf{P} + \overset{0}{\rho}\mathbf{f} = 0$, where $\mathbf{f}$ is specific mass forces. Differentiate the equation with respect to $\xi$: $\overset{0}{\nabla} \cdot \mathbf{P}_\xi = 0$. Variations of the specific mass forces' vector $\mathbf{f}$, and also of external surface forces' vector $\mathbf{S}^e$ and given displacements' vector $\mathbf{u}^e$, are assumed to be zero. Then we obtain the following equation in terms of the variations' vector $\mathbf{w}$:

$$\overset{0}{\nabla} \cdot \left( ({}^6\overset{(n)}{\mathbf{R}}{}^{0T} \cdot\cdot \overset{(n)}{\mathbf{T}} + {}^4\overset{(n)}{\mathbf{H}}{}^{(s)} \cdot\cdot {}^4\overset{(n)}{\mathbf{U}}) \cdot\cdot \boldsymbol{\varepsilon}(\mathbf{w}) + {}^6\overset{(n)}{\mathbf{R}}{}^{0T} \cdot\cdot \overset{(n)}{\mathbf{T}} \cdot\cdot \tilde{\boldsymbol{\varepsilon}}(\mathbf{w}) \right) = 0, \quad (17)$$

where ${}^6\overset{(n)}{\mathbf{R}}{}^{0T}$ are the transpose sixth-order tensors: ${}^6\overset{(n)}{\mathbf{R}}{}^{0T} = ({}^6\overset{(n)}{\mathbf{R}}{}^{0T})^{(125634)}$. Linear strain tensors $\boldsymbol{\varepsilon}(\mathbf{w})$ and $\tilde{\boldsymbol{\varepsilon}}(\mathbf{w})$ in the reference configuration basis have the form

$$\boldsymbol{\varepsilon}(\mathbf{w}) = \frac{1}{2}(\mathbf{F}^{-1\mathrm{T}} \cdot \overset{0}{\nabla} \otimes \mathbf{w} + \overset{0}{\nabla} \otimes \mathbf{w}^{\mathrm{T}} \cdot \mathbf{F}^{-1}), \quad \tilde{\boldsymbol{\varepsilon}}(\mathbf{w}) = \frac{1}{2}(\mathbf{F} \cdot \overset{0}{\nabla} \otimes \mathbf{w} + \overset{0}{\nabla} \otimes \mathbf{w}^{\mathrm{T}} \cdot \mathbf{F}^{\mathrm{T}}). \quad (18)$$

For the considered elastic body, boundary conditions are assumed to be given as the force vector $\mathbf{S}$ at surface part $\overset{0}{\Sigma}_\sigma$ and the displacements' vector $\mathbf{u}^e$ at part $\overset{0}{\Sigma}_u$:

$\overset{0}{\mathbf{n}}\cdot\mathbf{P}\Big|_{\overset{0}{\Sigma}_\sigma} = \mathbf{S}^e, \quad \mathbf{u}\Big|_{\overset{0}{\Sigma}_u} = \mathbf{u}^e$. Differentiating equations (15) and using expression (1) for variation of the displacements' vector $\hat{\mathbf{u}} = \hat{\mathbf{x}} - \overset{0}{\mathbf{x}} = \mathbf{x} + \xi\mathbf{w} - \overset{0}{\mathbf{x}} = \mathbf{u} + \xi\mathbf{w}$ and $\hat{\mathbf{u}}\Big|_{\overset{0}{\Sigma}_u} = \mathbf{u}^e$, we get $\overset{0}{\mathbf{n}}\cdot\mathbf{P}\Big|_{\overset{0}{\Sigma}_\sigma} = 0,\ \mathbf{w}\Big|_{\overset{0}{\Sigma}_u} = 0$. Substituting formulae (16) into these equations, we can write the boundary conditions in the form

$$\overset{0}{\mathbf{n}}\cdot\left( ({}^6\overset{(n)}{\mathbf{R}}{}^{0T}\cdot\cdot\overset{(n)}{\mathbf{T}} + {}^4\overset{(n)}{\mathbf{H}}{}^{(s)}\cdot\cdot{}^4\overset{(n)}{\mathbf{U}})\cdot\cdot\boldsymbol{\varepsilon}(\mathbf{w}) + {}^6\overset{(n)}{\mathbf{R}}{}^{0T}\cdot\cdot\overset{(n)}{\mathbf{T}}\cdot\cdot\tilde{\boldsymbol{\varepsilon}}(\mathbf{w}) \right)\Bigg|_{\overset{0}{\Sigma}_\sigma} = 0, \quad \mathbf{w}\Big|_{\overset{0}{\Sigma}_u} = 0. \tag{19}$$

The equations' system (17), (19) is the desired stability problem statement for a nonlinear-elastic body. This problem is linear with respect to the unknown functions' vector $\mathbf{w}$, includes derivatives of the second order and is uniform, i.e. admits the trivial solution $\mathbf{w} \equiv 0$. Solutions of the stability problem are just nontrivial solutions: $\mathbf{w} \neq 0$. The trivial solution corresponds to a stable equilibrium of the body, and nontrivial – to the nonstable one.

The stability problem statement (17), (19) is concerned to the class of problems on eigenvalues. Together with the problem (17), (19), let us consider the initial problem on body equilibrium in $K$ in Lagrangian description:

$$\overset{0}{\nabla}\cdot\mathbf{P} = 0, \quad \mathbf{P} = {}^4\overset{(n)}{\mathbf{E}}{}^0\cdot\cdot\overset{(n)}{\mathbf{T}}, \quad \overset{(n)}{\mathbf{T}} = \sum_{\gamma=1}^{\mathrm{r}} \varphi_\gamma \mathrm{I}^{(s)}_{\gamma\mathrm{C}}(\overset{(n)}{\mathbf{C}}), \quad \overset{(n)}{\mathbf{C}} = \frac{1}{\mathrm{n}-\mathrm{III}}(\mathbf{U}^{\mathrm{n}-\mathrm{III}} - \mathbf{E}),$$

$$\mathbf{U}^2 = \mathbf{F}^{\mathrm{T}}\cdot\mathbf{F}, \quad \mathbf{F} = \mathbf{E} + \overset{0}{\nabla}\otimes\mathbf{u}^{\mathrm{T}}, \quad \overset{0}{\mathbf{n}}\cdot\mathbf{P}\Big|_{\overset{0}{\Sigma}_\sigma} = \mu\overset{0}{\mathbf{S}}{}^e, \quad \mathbf{u}\Big|_{\overset{0}{\Sigma}_u} = \mu\overset{0}{\mathbf{u}}{}^e. \tag{20}$$

Here we introduced the scalar parameter $\mu$ being a multiplier at vectors of external surface forces $\mathbf{S}^e$ and displacements $\mathbf{u}^e$. Let us assume that a solution of the problem for the displacement vector $\mathbf{u}$ is found for values of $\mu$ from a certain interval $(\mu_1, \mu_2)$, then functions $\mathbf{u}(\mu)$ and $\overset{(n)}{\mathbf{T}}(\mu)$ may be considered. Substitute the stress tensor $\overset{(n)}{\mathbf{T}}(\mu)$ into the stability problem (17), (19) and include the parameter $\mu$ into the number of unknowns of the problem together with $\mathbf{w}$. Then the problem (17), (19) is formulated as follows: one should find such values of $\mu$, that equation system (17), (19) has the trivial solution $\mathbf{w}$. So it is the problem on eigenvalues. This problem is solved together with the main problem (20) of the nonlinear elasticity theory, because equations (17), (19) contain tensor $\overset{(n)}{\mathbf{T}}(\mu)$, which is a solution of the problem (20).

## Stability equations for models $A_{\mathrm{I}}$ and $A_{\mathrm{V}}$

Formulae (15) have the simplest form for model $A_{\mathrm{V}}$: $\mathbf{T} = \mathbf{F}\cdot\overset{V}{\mathbf{T}}\cdot\mathbf{F}^{\mathrm{T}}$, $\mathbf{P} = \sqrt{g/\overset{0}{g}}\,\overset{V}{\mathbf{T}}\cdot\mathbf{F}^{\mathrm{T}}$. On differentiating these equations with respect to $\xi$ and taking

(3) and (14) into account, the problem (17), (19) for $A_{\mathrm{V}}$ of a nonlinear-elastic solid takes the form

$$\overset{0}{\nabla}\cdot\left(({}^4\overset{V}{\mathbf{H}}{}^{(s)}\cdot\cdot\overset{0}{\boldsymbol{\varepsilon}})\cdot\mathbf{F}^{\mathrm{T}}+\overset{V}{\mathbf{T}}\cdot\overset{0}{\nabla}\otimes\mathbf{w}+(\mathbf{F}^{-1\mathrm{T}}\cdot\cdot\overset{0}{\nabla}\otimes\mathbf{w})\overset{V}{\mathbf{T}}\cdot\mathbf{F}^{\mathrm{T}}\right)=0$$

$$\overset{0}{\boldsymbol{\varepsilon}}=\frac{1}{2}(\overset{0}{\nabla}\otimes\mathbf{w}\cdot\mathbf{F}+\mathbf{F}^{\mathrm{T}}\cdot\overset{0}{\nabla}\otimes\mathbf{w}^{\mathrm{T}})$$

$$\overset{0}{\mathbf{n}}\cdot\left(({}^4\overset{V}{\mathbf{H}}{}^{(s)}\cdot\cdot\overset{0}{\boldsymbol{\varepsilon}})\cdot\mathbf{F}^{\mathrm{T}}+\overset{V}{\mathbf{T}}\cdot\overset{0}{\nabla}\otimes\mathbf{w}+(\mathbf{F}^{-1\mathrm{T}}\cdot\cdot\overset{0}{\nabla}\otimes\mathbf{w})\overset{V}{\mathbf{T}}\cdot\mathbf{F}^{\mathrm{T}}\right)\Bigg|_{\overset{0}{\Sigma}_{\sigma}}=0,\ \mathbf{w}\Big|_{\overset{o}{\Sigma}_u}=0. \tag{21}$$

For $A_{\mathrm{I}}$, formulae (15) give $\mathbf{T}=\mathbf{F}^{-1\mathrm{T}}\cdot\overset{I}{\mathbf{T}}\cdot\mathbf{F}^{-1}$, $\mathbf{P}=\sqrt{g/\overset{0}{g}}\mathbf{U}^{-2}\cdot\overset{I}{\mathbf{T}}\cdot\mathbf{F}^{-1}$. Differentiating these equations with respect to $\xi$, we get

$$\mathbf{T}_{\xi}=\mathbf{F}^{-1\mathrm{T}}\cdot\overset{I}{\mathbf{T}}_{\xi}\cdot\mathbf{F}^{-1}-\nabla\otimes\mathbf{w}\cdot\mathbf{F}^{-1\mathrm{T}}\cdot\overset{I}{\mathbf{T}}\cdot\mathbf{F}^{-1}-\mathbf{F}^{-1\mathrm{T}}\cdot\overset{I}{\mathbf{T}}\cdot\mathbf{F}^{-1}\cdot\nabla\otimes\mathbf{w}^{\mathrm{T}}, \tag{22}$$

$$\mathbf{P}_{\xi}=\sqrt{g/\overset{0}{g}}(\mathbf{U}^{-2}\cdot\overset{I}{\mathbf{T}}_{\xi}\cdot\mathbf{F}^{-1}-2\mathbf{F}^{-1}\cdot\boldsymbol{\varepsilon}\cdot\mathbf{F}^{-1\mathrm{T}}\cdot\overset{I}{\mathbf{T}}\cdot\mathbf{F}^{-1}-\mathbf{U}^{-2}\cdot\overset{I}{\mathbf{T}}\cdot\mathbf{F}^{-1}\cdot\nabla\otimes\mathbf{w}^{\mathrm{T}}+(\nabla\cdot\mathbf{w})\mathbf{U}^{-2}\cdot\overset{I}{\mathbf{T}}\cdot\mathbf{F}^{-1})$$

Then system (17), (19) for model $A_{\mathrm{I}}$ of a nonlinear-elastic solid takes the form

$$\overset{0}{\nabla}\cdot(({}^4\overset{I}{\mathbf{H}}{}^{(s)}\cdot\cdot\overset{0}{\tilde{\boldsymbol{\varepsilon}}})\cdot\mathbf{F}^{-1}-2\overset{0}{\tilde{\boldsymbol{\varepsilon}}}\cdot\overset{I}{\mathbf{T}}\cdot\mathbf{F}^{-1}-\mathbf{U}^{-2}\cdot\overset{I}{\mathbf{T}}\cdot\mathbf{U}^{-2}\cdot\overset{0}{\nabla}\otimes\mathbf{w}^{\mathrm{T}}+(\mathbf{F}^{-1\mathrm{T}}\cdot\cdot\overset{0}{\nabla}\otimes\mathbf{w})\mathbf{U}^{-2}\cdot\overset{I}{\mathbf{T}}\cdot\mathbf{F}^{-1})=0$$

$$\overset{0}{\tilde{\boldsymbol{\varepsilon}}}(\mathbf{w})=\frac{1}{2}(\mathbf{U}^{-2}\cdot\overset{0}{\nabla}\otimes\mathbf{w}\cdot\mathbf{F}^{-1T}+\mathbf{F}^{-1}\cdot\overset{0}{\nabla}\otimes\mathbf{w}^{\mathrm{T}}\cdot\mathbf{U}^{-2}),\quad \mathbf{w}\Big|_{\overset{0}{\Sigma}_u}=0, \tag{23}$$

$$\overset{0}{\mathbf{n}}\cdot(({}^4\overset{I}{\mathbf{H}}{}^{(s)}\cdot\cdot\overset{0}{\tilde{\boldsymbol{\varepsilon}}})\cdot\mathbf{F}^{-1}-2\overset{0}{\tilde{\boldsymbol{\varepsilon}}}\cdot\overset{I}{\mathbf{T}}\cdot\mathbf{F}^{-1}-\mathbf{U}^{-2}\cdot\overset{I}{\mathbf{T}}\cdot\mathbf{U}^{-2}\cdot\overset{0}{\nabla}\otimes\mathbf{w}^{\mathrm{T}}+(\mathbf{F}^{-1\mathrm{T}}\cdot\cdot\overset{0}{\nabla}\otimes\mathbf{w})\mathbf{U}^{-2}\cdot\overset{I}{\mathbf{T}}\cdot\mathbf{F}^{-1})_{\overset{0}{\Sigma}_{\sigma}}=0$$

Stress tensors $\overset{I}{\mathbf{T}}$ and $\overset{V}{\mathbf{T}}$ in problems (21), (23) are determined by solving the problem on a basic state equilibrium (22). For other models ($A_{\mathrm{II}}$, $A_{\mathrm{IV}}$) there is a need to use the general equations (17), (19). Comparing the derived equation systems (21), (23), and the general system (17), (19) for different values *n*, we get that the linearized systems of stability theory equations prove different for different models of the nonlinear-elastic behavior of a solid. Thus, ultimate external loads leading to the loss of solids' stability will be different as well.

## Conclusions

The generalized three-dimensional stability theory of nonlinear-elastic solids is suggested for the case of arbitrary finite deformations, which is based on the concept of a varied configuration of nonlinear-elastic solids and with use of the generalized models of nonlinear-elastic solids developed by the author with the help of five energetic couples of stress-strain tensors. The final equations of the three-dimensional stability theory prove to be different for different models of nonlinear-elastic solids. The explicit analytical equations of the stability theory are found for two types of the models including the right Almansi and Cauchy-Green strain tensors, when there is no need to calculate eigenvalues of the stretch tensor. The derived equations of the three-dimensional stability theory have the universal character, i.e. they may be applied to calculations of stability of complicated nonlinear-elastic solids in frames of three-dimensional analysis of a stress-strain state as well as solids with small elastic deformations, and also for calculating in frames of two-dimensional shell structures.

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