Some Consideration with Respect to the Relation between Transform of Derivative and Differentiation of Transform

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Abstract
The relation between transform of derivative and differentiation of transform has been checked. The relation plays an important role to represent differential equations with variable coefficients.

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1 Introduction
Integral transforms are an useful tool for solving linear ODEs and corresponding initial value problems, and this method gives an probable algebraic repre-
sentation to solve various engineering problems. In connection with this, several researches have been pursued[2-6, 8-15, 17-23] such as Fourier, Sumudu, Elzaki transform, and so on. Among these things, Laplace transform is the representative thing, but the Laplace transform method has a weak point in applying to ODEs with variable coefficients because the representation of $F^{(n)}(s)$ is not suitable. For example, $F''(s) = ds^2/dt^2 = d/ds\{-\mathcal{L}(tf(t))\} = \mathcal{L}(t^2f(t))$, and it is very difficult to handle this term. Let us see a basic Euler-Cauchy equation $t^2y'' + aty' + by = 0$. Taking Laplace transform on both sides, we have

$$\frac{d}{ds}(-2sY - s^2 \frac{dY}{ds} + y(0)) - a(Y + s \frac{dY}{ds}) + bY = 0,$$

where $\mathcal{L}(y) = F(s) = Y$. Organizing this equality[12], we have

$$S^2 \frac{d^2Y}{ds^2} + (4 - a)s \frac{dY}{ds} + (b - a + 2)Y = 0.$$

Here, although the solution $y$ can be represented by $s$, it is too complicated. Hence, we need to think the relation between $\mathcal{L}(y')$ and $F'(s)$, because it is a key in the representation of differential equations with variable coefficients by using integral transforms. In other word, if $F''(s)$ can be represented by terms of $\mathcal{L}(y'')$, it will be very attractive and effective thing in transform theories. Of course, $\mathcal{L}(y')$ means the transform of derivative, and $F'(s)$ means the differentiation of transform. To begin with, let us recall the relation $\mathcal{L}(y) = F(s)$ but $\mathcal{L}(y') \neq F'(s)$. For example, let us consider the case of $y = y(t) = t$. Then $\mathcal{L}(t) = 1/s^2$, $\mathcal{L}(y') = 1/s$ but $F'(s) = -2/s^3$. Thus, we can easily know that $\mathcal{L}(y') \neq F'(s)$.

There is no previous research which be directly involved in this topic. Hence, we would like to get to the first base. In this article, we have checked several expressions with respect to the relation of $\mathcal{L}(y')$ and $F'(s)$, and applied properties of measure theory to the relation.

\section{The relation of $\mathcal{L}(y')$ and $F'(s)$}

We would like to check the relation of $\mathcal{L}(y')$ and $F'(s)$. To begin with, let us see the definition of summable function. A measurable function $f$ is said to be summable if the integral of the absolute value of $f$ exists and is finite, that is, $\int |f|dt < \infty$. An summable function is also called an absolutely integrable one.

\textbf{Lemma 2.1} Let $(X, A, \mu)$ be a measure space, and let $f$ be a $[-\infty, \infty]$-valued $A$-measurable function on $X$. Then $f$ is integralbe if and only if $|f|$ is integrable. if these functions are integrable, then $\int |f|d\mu \leq \int |f|d\mu[1, 7]$. 
In general, the Laplace transform $\mathcal{L}(f)$ exists for all $s > k$ if $f(t)$ is defined and piecewise continuous on $t \geq 0$ and satisfies $|f(t)| \leq Me^{kt}$. Since piecewise continuous function is measurable, the above lemma is still valid for Laplace transform as appearing in the following theorem.

**Theorem 2.2** Let $\mathcal{L}(y) = F(s)$ be the Laplace transform of $y(t)$. Then the following statements are established.

1. The function $\mathcal{L}(y') - F'(s)$ and $y' + ty$ are summable.
2. $\mathcal{L}(y' + ty) = \mathcal{L}(y') - F'(s)$.
3. $|\mathcal{L}(y')F'(s)| + \mathcal{L}(y')F'(s) \geq 0$ and $|F'(s)| + F'(s) \geq 0$.

**Proof.**

1. Since the Laplace transform of $y(t)$ is $F(s) = \mathcal{L}(y) = \int_0^\infty e^{-st}y(t)\,dt$, the difference of $\mathcal{L}(y')$ and $F'(s)$ can be expressed as
   
   $$|\mathcal{L}(y') - F'(s)| = \left| \int_0^\infty e^{-st}(y'(t) + ty(t))\,dt \right| \leq \int_0^\infty |e^{-st}||y' + ty|\,dt$$

   because of lemma 2.1. Since $|e^{-st}| \leq 1$ on $t \geq 0$, we have
   
   $$|\mathcal{L}(y') - F'(s)| \leq \int_0^\infty |y' + ty|\,dt < \infty.$$

   Where, the integrability of $\mathcal{L}(y') - F'(s)$ is followed from that of $\mathcal{L}(y')$ and $F'(s)$.

2. $\mathcal{L}(y' + ty) = \mathcal{L}(y') + \mathcal{L}(ty) = \mathcal{L}(y') - F'(s)$.

3. From $|\mathcal{L}(y')| + |F'(s)| \geq |\mathcal{L}(y') + F'(s)| \geq |\mathcal{L}(y') - F'(s)|$, we have
   
   $$|\mathcal{L}(y')F'(s)| + \mathcal{L}(y')F'(s) \geq 0.$$

Since $\mathcal{L}(y') \geq 0$, we have $\mathcal{L}(y')(|F'(s)| + F'(s)) \geq 0$.

From the direct calculation, we easily get the equation $F'(s) = -\int_0^\infty e^{-st}tf(t)\,dt = -\mathcal{L}[tf(t)]$. The result can be checked from the above (2).

$$F'(s) = \mathcal{L}(y') - \mathcal{L}(y' + ty) = -\mathcal{L}(ty).$$

**Example 2.3** In $F(s) = \int_0^\infty e^{-st}f(t)\,dt$, we must assume that the integral exists.

**Solution.** Using $|\mathcal{L}(y') - F'(s)|$, let us check this. Since the complex number can be expressed by the exponential function, let us put $y = e^{it}$. Then

$$|\mathcal{L}(y') - F'(s)| \leq \int_0^\infty |y' + ty|\,dt = \int_0^\infty |e^{it}||i + t|\,dt$$

$$\leq \int_0^\infty |i|\,dt + \int_0^\infty |t|\,dt = \int_0^\infty dt + \int_0^\infty |t|\,dt = \infty$$

because of $|e^{it}| = |\cos t + is\sin t| = 1$. 

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*The relation of $\mathcal{L}(y')$ and $F'(s)$*
Theorem 2.4 Let \( n \) be an arbitrary natural number. Then the difference of \( \mathcal{L}(y') \) and \( F'(s) \) can be represented as

\[
\sum_{k=1}^{n} \frac{1}{s^k} [y^{(k)}(0) + (k-1)y^{(k-2)}(0)] + \frac{1}{s^n} \mathcal{L}[y^{(n+1)} + ty^{(n)} + ny^{(n-1)}]
\]

for \( y^{(k)} \) is the \( k \)-th derivative of a given function \( y(t) \).

Proof. We would like to establish the validity of the statement by the mathematical induction, and note that the difference of \( \mathcal{L}(y') \) and \( F'(s) \) equals to \( \mathcal{L}(y' + ty) \). For \( k = 1 \), by integration by parts,

\[
\mathcal{L}(y' + ty) = \int_{0}^{\infty} e^{-st}(y' + ty) \, dt
\]

\[
= -\frac{1}{s} e^{-st}(y' + ty)|_{0}^{\infty} + \frac{1}{s} \int_{0}^{\infty} e^{-st}(y'' + ty' + y) \, dt
\]

\[
= \frac{1}{s} y'(0) + \frac{1}{s} \mathcal{L}(y'' + ty' + y)
\]

holds for \( y = y(t) \). Next, we suppose that

\[
\mathcal{L}(y' + ty) = \sum_{k=1}^{m} \frac{1}{s^k} [y^{(k)}(0) + (k-1)y^{(k-2)}(0)] + \frac{1}{s^m} \mathcal{L}[y^{(m+1)} + ty^{(m)} + my^{(m-1)}], \quad (*)
\]

and show that \( \mathcal{L}(y' + ty) \) can be expressed by

\[
\sum_{k=1}^{m+1} \frac{1}{s^k} [y^{(k)}(0) + (k-1)y^{(k-2)}(0)] + \frac{1}{s^{m+1}} \mathcal{L}[y^{(m+2)} + ty^{(m+1)} + (m+1)y^{(m)}].
\]

From (*),

\[
\mathcal{L}(y' + ty) = \frac{1}{s} y'(0) + \frac{1}{s^2} [y''(0) + y(0)]
\]

\[
+ \frac{1}{s^3} [y^3(0) + 2y'(0)] + \cdots + \frac{1}{s^m} [y^m(0) + (m-1)y^{(m-2)}(0)]
\]

\[
+ \frac{1}{s^m} \mathcal{L}[y^{(m+1)} + ty^{(m)} + my^{(m-1)}].
\]

Now, let us consider the transform

\[
\mathcal{L}[y^{(m+1)} + ty^{(m)} + my^{(m-1)}]. \quad (**)
\]

Since (**) equal to

\[
= \int_{0}^{\infty} e^{-st}[y^{(m+1)} + ty^{(m)} + my^{(m-1)}] \, dt
\]
The relation of \( L(y') \) and \( F'(s) \)

\[
\begin{align*}
&= -\frac{1}{s} e^{-st}[y^{(m+1)} + ty^{(m)} + my^{(m-1)}]_0^\infty \\
&+ \frac{1}{s} \int_0^\infty e^{-st}[y^{(m+2)} + ty^{(m+1)} + (m + 1)y^{(m)}] \\
&= \frac{1}{s} [y^{(m+1)}(0) + my^{(m-1)}(0)] \\
&+ \frac{1}{s} L[y^{(m+2)} + ty^{(m+1)} + (m + 1)y^{(m)}] \\
&= \frac{1}{s} \sum_{k=1}^{m+1} \frac{1}{s^k} [y^{(k)}(0) + (k - 1)y^{(k-2)}(0)] \\
&+ \frac{1}{s^{m+1}} L[y^{(m+2)} + ty^{(m+1)} + (m + 1)y^{(m)}].
\end{align*}
\]

for \( y = y(t) \). Here, we note that \( (ty^{(m)})' = y^{(m)} + ty^{(m+1)} \). Hence,

\[
L(y' + ty) = \sum_{k=1}^{m} \frac{1}{s^k} [y^{(k)}(0) + (k - 1)y^{(k-2)}(0)] \\
+ \frac{1}{s^{m+1}} [y^{(m+1)}(0) + my^{(m-1)}(0)] \\
+ \frac{1}{s^{m+1}} L[y^{(m+2)} + ty^{(m+1)} + (m + 1)y^{(m)}] \\
= \sum_{k=1}^{m+1} \frac{1}{s^k} [y^{(k)}(0) + (k - 1)y^{(k-2)}(0)] \\
+ \frac{1}{s^{m+1}} L[y^{(m+2)} + ty^{(m+1)} + (m + 1)y^{(m)}].
\]

Thus, if the equality holds for \( k \), it holds for \( k+1 \). Therefore, by mathematical induction, the equality is true for all natural number \( n \). The proof is completed.

References


The relation of $\mathcal{L}(y')$ and $F'(s)$


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