Generalized Extended Whittaker Function and Its Properties

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Abstract

We introduce a further generalization of the extended Whittaker function by using the generalized extended confluent hypergeometric function of the first kind and investigate, in a rather systematic manner, its integral representations, some integral transforms, differential formula and recurrence relations. Relevant connections of some results presented here with those involving relatively simpler known formulas are also indicated. In view of diverse applications of the Whittaker function in the mathematical physics, the results here may be potentially useful in some related research areas.

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1 Introduction and Preliminaries

The Whittaker function $M_{k,\mu}(z)$ is defined in terms of confluent hypergeometric function (or Kummer’s function) of first kind $\Phi(\cdot)$ as follows (see, e.g., [11, 12, 13]):

$$M_{k,\mu}(z) = z^{\mu + \frac{1}{2}} \exp\left(-\frac{z}{2}\right) \Phi\left(\mu - k + \frac{1}{2}; 2\mu + 1; z\right) \quad \left(\Re(\mu) > -\frac{1}{2}, \Re(\mu \pm k) > -\frac{1}{2}\right).$$

(1)

Very recently, Parmar [7] introduced and investigated some fundamental properties and characteristics of more generalized beta type function $B_{\sigma}(\alpha,\beta;m)$ defined by

$$B_{\sigma}(\alpha,\beta;m)(x,y) = \int_{0}^{1} t^{x-1} (1-t)^{y-1} \mathrm{I}_{1} F_{1}\left(\alpha; \beta; -\frac{\sigma}{t^{m}(1-t)^{m}}\right) dt \quad (\Re(\sigma) > 0, \Re(x) > 0, \Re(y) > 0, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(m) > 0).$$

(2)

The special case of (2) when $m = 1$ reduces to the known generalized beta type function (see Özergin et al. [6]):

$$B_{\sigma}(\alpha,\beta)(x,y) = \int_{0}^{1} t^{x-1} (1-t)^{y-1} \exp\left(-\frac{\sigma}{t(1-t)}\right) dt \quad (\Re(\sigma) > 0, \Re(x) > 0, \Re(y) > 0, \Re(\alpha) > 0, \Re(\beta) > 0).$$

(3)

Chaudhry et al. [1] introduced and investigated the case $\alpha = \beta$ of (3):

$$B_{\sigma}(x,y) = B_{\sigma}(\alpha,\alpha)(x,y) = \int_{0}^{1} t^{x-1} (1-t)^{y-1} \exp\left(-\frac{\sigma}{t(1-t)}\right) dt \quad (\Re(\sigma) > 0).$$

(4)

It is easy to see that (4) reduces the classical beta function $B(x,y)$ (see, e.g., [10, Section 1.1]) as follows:

$$B(x,y) = B_{0}(x,y) = B_{0}(\alpha,\beta)(x,y).$$

Using (4), Chaudhry et al. [2] extended the Gauss hypergeometric function and the confluent hypergeometric function, respectively, as follows:

$$F_{\sigma}(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_{n}}{B(b,c-b)} \frac{B_{\sigma}(b+n, c-b) z^{n}}{n!} \quad (\sigma \geq 0, |z| < 1, \Re(c) > \Re(b) > 0)$$

(5)

and

$$\Phi_{\sigma}(b;c;z) = \sum_{n=0}^{\infty} \frac{B_{\sigma}(b+n, c-b)}{B(b,c-b)} \frac{z^{n}}{n!} \quad (\sigma \geq 0, \Re(c) > \Re(b) > 0).$$

(6)
Among several interesting formulas given in [2], the following Euler's type integral representations are recalled:

\[
F_\sigma(a, b; c; z) = \frac{1}{B(b, c - b)} \int_0^1 t^{b-1} (1 - t)^{c-b-1} \left(1 - zt\right)^{-a} \exp \left[-\frac{\sigma}{t(1-t)}\right] dt \tag{7}
\]

\(\sigma > 0; \ \sigma = 0, \ |\arg(1 - z)| < \pi, \ \Re(c) > \Re(b) > 0\)

and

\[
\Phi_\sigma(b; c; z) = \frac{1}{B(b, c - b)} \int_0^1 t^{b-1} (1 - t)^{c-b-1} \exp \left[zt - \frac{\sigma}{t(1-t)}\right] dt \tag{8}
\]

\(\sigma > 0; \ \sigma = 0, \ \Re(c) > \Re(b) > 0\).

By appealing to \(B^{(\alpha, \beta)}_\sigma(x, y)\) in (3), Özergin et al. [6] further extended the Gauss hypergeometric function and the confluent hypergeometric function, respectively, as follows:

\[
F^{(\alpha, \beta)}_\sigma(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n B^{(\alpha, \beta)}_\sigma(b + n, c - b) z^n}{B(b, c - b) n!} \tag{9}
\]

\(\sigma \geq 0, \ |z| < 1, \ \Re(c) > \Re(b) > 0, \ \Re(\alpha) > 0, \ \Re(\beta) > 0\)

and

\[
\Phi^{(\alpha, \beta)}_\sigma(b; c; z) = \sum_{n=0}^{\infty} \frac{B^{(\alpha, \beta)}_\sigma(b + n, c - b) z^n}{B(b, c - b) n!} \tag{10}
\]

\(\sigma \geq 0, \ \Re(c) > \Re(b) > 0, \ \Re(\alpha) > 0, \ \Re(\beta) > 0\)

with their respective Eulerian type integral representations

\[
F^{(\alpha, \beta)}_\sigma(a, b; c; z) = \frac{1}{B(b, c - b)} \int_0^1 t^{b-1} (1 - t)^{c-b-1} \times (1 - zt)^{-a} {}_1F_1 \left(\alpha; \beta; -\frac{\sigma}{t(1-t)}\right) dt \tag{11}
\]

\(\sigma > 0, \ \sigma = 0, \ |\arg(1 - z)| < \pi, \ \Re(c) > \Re(b) > 0, \ \Re(\alpha) > 0, \ \Re(\beta) > 0\)

and

\[
\Phi^{(\alpha, \beta)}_\sigma(b; c; z) = \frac{1}{B(b, c - b)} \int_0^1 t^{b-1} (1 - t)^{c-b-1} e^{zt} {}_1F_1 \left(\alpha; \beta; -\frac{\sigma}{t(1-t)}\right) dt \tag{12}
\]

\(\sigma > 0, \ \sigma = 0, \ \Re(c) > \Re(b) > 0, \ \Re(\alpha) > 0, \ \Re(\beta) > 0\).
By using $B_{a,\beta}^{(\sigma)}(x, y)$ in (2), Parmar [7] presented further generalizations of the extended Gauss hypergeometric function and the confluent hypergeometric function, respectively, as follows:

$$F_{a,\beta}^{(\sigma)}(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n B_{a,\beta}^{(\sigma)}(b + n, c - b) z^n}{B(b, c - b) n!}$$

$$\text{for } (\sigma \geq 0; \ |z| < 1, \ \Re(c) > \Re(b) > 0, \ \Re(\alpha) > 0, \ \Re(\beta) > 0, \ \Re(m) > 0)$$

and

$$\Phi_{a,\beta}^{(\sigma)}(a, b; c; z) = \sum_{n=0}^{\infty} \frac{B_{a,\beta}^{(\sigma)}(b + n, c - b) z^n}{B(b, c - b) n!}$$

$$\text{for } (\sigma \geq 0, \ \Re(c) > \Re(b) > 0, \ \Re(\alpha) > 0, \ \Re(\beta) > 0, \ \Re(m) > 0)$$

with their respective Eulerian type integral representations:

$$F_{a,\beta}^{(\sigma)}(a, b; c; z) = \frac{1}{B(b, c - b)} \int_0^1 t^{b-1} (1 - t)^{c-b-1} (1 - zt)^{-\sigma}$$

$$\times {}_1F_1 \left( \alpha; \beta; -\frac{\sigma}{t^m(1-t)^m} \right) dt$$

$$\text{for } (\sigma > 0, \ \Re(c) > \Re(b) > 0, \ \Re(m) > 0)$$

and

$$\Phi_{a,\beta}^{(\sigma)}(a, b; c; z) = \frac{1}{B(b, c - b)} \int_0^1 t^{b-1} (1 - t)^{c-b-1} e^{zt}$$

$$\times {}_1F_1 \left( \alpha; \beta; -\frac{\sigma}{t^m(1-t)^m} \right) dt$$

$$\text{for } (\sigma > 0, \ \Re(c) > \Re(b) > 0, \ \Re(m) > 0).$$

Setting $t = 1 - u$ in (16), Parmar [7] obtained the following relation for the generalized extended confluent hypergeometric function of the first kind:

$$\Phi_{a,\beta}^{(\sigma)}(b; c; z) = \exp(z) \ \Phi_{a,\beta}^{(\sigma)}(c - b; c; -z),$$

whose special case $\sigma = 0$ reduces to the Kummer’s first formula for the classical confluent hypergeometric function [8, Theorem 42, p. 125].

## 2 Generalized extended Whittaker function

Here we give a generalized extended Whittaker function in terms of the generalized extended confluent hypergeometric function. Some integral representations and a relation for this function are also derived.
**Definition 2.1.** A generalized extended Whittaker function denoted by \( M_{\sigma,k,\mu}^{(\alpha,\beta;m)}(z) \) is defined by
\[
M_{\sigma,k,\mu}^{(\alpha,\beta;m)}(z) = z^{\mu + \frac{1}{2}} \exp\left(-\frac{z}{2}\right) \Phi_{\sigma}^{(\alpha,\beta;m)}(\mu - k + \frac{1}{2}; 2\mu + 1; z)
\]
where \( \Phi_{\sigma}^{(\alpha,\beta;m)} \) is the generalized extended confluent hypergeometric function of the first kind in (16).

**Remark 2.2.** The special case of (18) when \( \alpha = \beta \) is seen to reduce to the generalized extended Whittaker function given by Khan and Ghayasuddin [4], the case \( m = 1 \) of which gives the extended Whittaker function introduced by Nagar et al. [5]. It is easy to see that \( \alpha = \beta, m = 1 \) and \( \sigma = 0 \), or, simply, \( \sigma = 0 \) of (18) yields the classical Whittaker function (1).

Certain interesting integral representations of the generalized extended Whittaker function \( M_{\sigma,k,\mu}^{(\alpha,\beta;m)}(z) \) in (18) are given in the following theorem.

**Theorem 2.3.** Suppose that
\[
\sigma > 0; \sigma = 0, \Re(\mu) > \Re(\mu \pm k) > -\frac{1}{2}, \text{ and } \Re(m) > 0.
\]
Each of the following integral formulas holds true:
\[
M_{\sigma,k,\mu}^{(\alpha,\beta;m)}(z) = z^{\mu + \frac{1}{2}} \exp\left(-\frac{z}{2}\right) \frac{1}{B(\mu - k + \frac{1}{2}, \mu + k + \frac{1}{2})} \times \int_0^1 t^{\mu-k-\frac{1}{2}} (1 - t)^{\mu+k-\frac{1}{2}} e^{zt} \, _1F_1\left(\alpha; \beta; -\frac{\sigma}{tm(1-t)^m}\right) \, dt;
\]
\[
M_{\sigma,k,\mu}^{(\alpha,\beta;m)}(z) = z^{\mu + \frac{1}{2}} \exp\left(-\frac{z}{2}\right) \frac{1}{B(\mu - k + \frac{1}{2}, \mu + k + \frac{1}{2})} \times \int_0^1 u^{\mu+k-\frac{1}{2}} (1 - u)^{\mu-k-\frac{1}{2}} e^{-zu} \, _1F_1\left(\alpha; \beta; -\frac{\sigma}{um(1-u)^m}\right) \, du;
\]
\[
M_{\sigma,k,\mu}^{(\alpha,\beta;m)}(z) = \frac{(q - p)^{-2\mu} z^{\mu + \frac{1}{2}} \exp\left(-\frac{z}{2}\right)}{B(\mu - k + \frac{1}{2}, \mu + k + \frac{1}{2})} \times \int_p^q (u - p)^{\mu-k-\frac{1}{2}} (q - u)^{\mu+k-\frac{1}{2}} \exp\left[\frac{z(u - p)}{(q - p)}\right] \, _1F_1\left(\alpha; \beta; -\frac{\sigma (q - p)2m}{(u - p)m(q - u)^m}\right) \, du.
\]
where \( p \) and \( q \) are two parameters such that \( q - p > 0 \);

\[
M^{(\alpha,\beta;m)}_{\sigma,k,\mu}(z) = \frac{\exp\left(-\frac{z}{2}\right) z^{\mu+\frac{1}{2}}}{B(\mu - k + \frac{1}{2}, \mu + k + \frac{1}{2})} \int_{0}^{\infty} u^{\mu-k-\frac{1}{2}} \times (1 + u)^{-(2\mu+1)} \exp\left(\frac{zu}{1+u}\right) \Phi_{\sigma}(\alpha; \beta; -\frac{\sigma (1+u)^{2m}}{u^{m}}) \, du;
\]

\[
M^{(\alpha,\beta;m)}_{\sigma,k,\mu}(z) = \frac{2^{-2\mu} z^{\mu+\frac{1}{2}}}{B(\mu - k + \frac{1}{2}, \mu + k + \frac{1}{2})} \int_{-1}^{1} (1 + u)^{\mu-k-\frac{1}{2}} \times (1 - u)^{\mu+k-\frac{1}{2}} \exp\left(\frac{zu}{2}\right) \Phi_{\sigma}(\alpha; \beta; -\frac{2^{2m} \sigma}{(1-u^{2})^{m}}) \, du.
\]

**Proof.** The use of (16) in (18) is seen to yield the integral representation (19). Setting \( t = 1 - u, t = \frac{u-p}{q-p}, \) and \( t = u^{1+u} \) in (19) yield (20), (21), and (22), respectively. Setting \( q = 1 \) and \( p = -1 \) in (21) gives (23).

**Remark 2.4.** Using (16) in the above expression (20), we get

\[
M^{(\alpha,\beta;m)}_{\sigma,k,\mu}(z) = z^{\mu+\frac{1}{2}} \exp\left(\frac{z}{2}\right) \Phi_{\sigma}(\alpha; \beta; (\mu + k + \frac{1}{2}; 2\mu + 1; z).
\]

Thus it is seen that the generalized extended Whittaker function can also be expressed by (24). The case \( \alpha = \beta \) of (19), (21), (22), and (23) is seen to yield those integral representations of the generalized extended Whittaker function given by Khan and Ghayasuddin [4], which, on further setting \( m = 1 \), correspond with the integral representations for the extended Whittaker function defined by Nagar et al. [5].

The generalized extended Whittaker function denoted by \( M^{(\alpha,\beta;m)}_{\sigma,k,\mu}(z) \) in (18) has an interesting relation asserted by Theorem 2.5.

**Theorem 2.5.** The following relation holds true:

\[
M^{(\alpha,\beta;m)}_{\sigma,k,\mu}(-z) = (-1)^{\mu+\frac{1}{2}} M^{(\alpha,\beta;m)}_{\sigma,-k,\mu}(z)
\]

\[
(\sigma \geq 0, \; m \geq 1, \; \Re(\alpha) > 0, \; \Re(\beta) > 0, \; \Re(\mu) > -\frac{1}{2}, \; \Re(\mu \pm k) > -\frac{1}{2}),
\]

**Proof.** Replacing \( z \) by \( -z \) in (18), we get

\[
M^{(\alpha,\beta;m)}_{\sigma,k,\mu}(-z) = (-z)^{\mu+\frac{1}{2}} \exp\left(\frac{z}{2}\right) \Phi_{\sigma}(\alpha; \beta; (\mu - k + \frac{1}{2}; 2\mu + 1; -z).
\]

Then applying (17) to (26) and writing the resulting expression in terms of (18), we get the desired result.
3 Integral transforms of $M^{(\alpha,\beta;m)}_{\sigma,k,\mu}(z)$

Theorem 3.1. The following Mellin transformation holds true:

$$\int_0^\infty \sigma^{s-1} M^{(\alpha,\beta;m)}_{\sigma,k,\mu}(z) \, d\sigma = \frac{\Gamma^{(\alpha,\beta)}(s) B(\mu - k + ms + \frac{1}{2}, \mu + k + ms + \frac{1}{2})}{z^{ms} B(\mu - k + \frac{1}{2}, \mu + k + \frac{1}{2})} M_{k,\mu+ms}(z) \quad (27)$$

where $M_{k,\mu+ms}$ is the classical Whittaker function.

Proof. Using the integral representation of $M^{(\alpha,\beta;m)}_{\sigma,k,\mu}(z)$ in (19) and changing the order of integration, we get

$$\int_0^\infty \sigma^{s-1} M^{(\alpha,\beta;m)}_{\sigma,k,\mu}(z) \, d\sigma = \frac{z^{\mu+\frac{1}{2}} \exp(-\frac{z}{2})}{B(\mu - k + \frac{1}{2}, \mu + k + \frac{1}{2})} \int_0^1 t^{\mu-k-\frac{1}{2}} (1-t)^{\mu-k-\frac{1}{2}} e^{zt} \left( \int_0^\infty \sigma^{s-1} 1_F(\alpha; \beta; -\frac{\sigma}{tm(1-t)^m}) \, d\sigma \right) \, dt.$$

Setting $u = \frac{\sigma}{tm(1-t)^m}$, we obtain

$$\int_0^\infty \sigma^{s-1} 1_F(\alpha; \beta; -\frac{\sigma}{tm(1-t)^m}) \, d\sigma = t^{ms} (1-t)^{ms} \int_0^\infty u^{s-1} 1_F(\alpha; \beta; -u) \, du = t^{ms} (1-t)^{ms} \Gamma^{(\alpha,\beta)}(s),$$

where $\Gamma^{(\alpha,\beta)}(s)$ is the generalized gamma function in [6].

We thus have

$$\int_0^\infty \sigma^{s-1} M^{(\alpha,\beta;m)}_{\sigma,k,\mu}(z) \, d\sigma = \frac{\Gamma^{(\alpha,\beta)}(s) z^{\mu+\frac{1}{2}} \exp(-\frac{z}{2})}{B(\mu - k + \frac{1}{2}, \mu + k + \frac{1}{2})} \int_0^1 t^{\mu+ms-k-\frac{1}{2}} (1-t)^{\mu+ms+k-\frac{1}{2}} e^{zt} \, dt = \frac{\Gamma^{(\alpha,\beta)}(s) B(\mu - k + ms + \frac{1}{2}, \mu + k + ms + \frac{1}{2})}{z^{ms} B(\mu - k + \frac{1}{2}, \mu + k + \frac{1}{2})} M_{k,\mu+ms}(z).$$

This completes the proof. \qed
Theorem 3.2. The following formula holds true:

\[
\int_0^\infty z^{a-1} e^{-pz} M_{\sigma,k,\mu}^{(\alpha,\beta;m)}(\beta z) \, dz = \frac{\beta^{\mu+\frac{1}{2}} \Gamma(a + \mu + \frac{1}{2})}{(p + \frac{\beta}{2})^{a+\mu+\frac{1}{2}}} \times \text{F}^{(\alpha,\beta;m)}_{\sigma}(a + \mu + \frac{1}{2}, \mu - k + \frac{1}{2}; 2\mu + 1; \frac{2\beta}{2p + \beta})
\]  

(28)

where \(\text{F}^{(\alpha,\beta;m)}_{\sigma}(a, b; c; z)\) is the generalized extended Gauss hypergeometric function in (15).

Proof. Using the integral representation of \(M_{\sigma,k,\mu}^{(\alpha,\beta;m)}(z)\) in (19), we get

\[
\int_0^\infty z^{a-1} e^{-pz} M_{\sigma,k,\mu}^{(\alpha,\beta;m)}(\beta z) \, dz = \frac{1}{B(\mu - k + \frac{1}{2}, \mu + k + \frac{1}{2})} \times \int_0^\infty z^{a-1} e^{-pz} (\beta z)^{\mu + \frac{1}{2}} e^{-\frac{\beta z}{2}} \times \int_0^1 t^{\mu-k-\frac{1}{2}}(1-t)^{\mu+k-\frac{1}{2}} e^{\beta z t} _1F_1\left(\alpha; \beta; -\frac{\sigma}{t^m(1-t)^m}\right) dt \, dz.
\]

Now changing the order of integration and integrating the resulting expression with respect to \(z\) by using the definition of Gamma function, we have

\[
\int_0^\infty z^{a-1} e^{-pz} M_{\sigma,k,\mu}^{(\alpha,\beta;m)}(\beta z) \, dz = \frac{\beta^{\mu+\frac{1}{2}} \Gamma(a + \mu + \frac{1}{2})}{(p + \frac{\beta}{2})^{a+\mu+\frac{1}{2}}} \int_0^1 t^{\mu-k-\frac{1}{2}}(1-t)^{\mu+k-\frac{1}{2}} \times \left(1 - \frac{2\beta t}{2p + \beta}\right)^{-(a+\mu+\frac{1}{2})} _1F_1\left(\alpha; \beta; -\frac{\sigma}{t^m(1-t)^m}\right) dt.
\]

(29)

Finally, applying (15) to (29) is seen to yield the desired result. \(\Box\)

The special case of (28) when \(\beta = a = 1\) yields a simpler integral formula given in Corollary 3.3.

Corollary 3.3. The following formula holds true:

\[
\int_0^\infty e^{-pz} M_{\sigma,k,\mu}^{(\alpha,\beta;m)}(z) \, dz = \frac{2^{\mu+\frac{3}{2}} \Gamma(\mu + \frac{3}{2})}{(2p + 1)^{\mu+\frac{3}{2}}} \times \text{F}^{(\alpha,\beta;m)}_{\sigma}(\mu + \frac{3}{2}, \mu - k + \frac{1}{2}; 2\mu + 1; \frac{2}{2p + 1})
\]

(30)

\[
\left(\sigma \geq 0, \ p > \frac{1}{2}, \ \Re(\mu) > -\frac{3}{2}\right).
\]
Theorem 3.4. The following Hankel transformation holds true:

\[
\int_0^\infty z M_{\sigma,k,\mu}^{(\alpha,\beta;m)}(z) J_\nu(az) \, dz = \frac{\Gamma(\mu + \nu + \frac{3}{2})}{(a^2 + \frac{1}{4})^{\frac{\mu + \nu + 1}{2}}} \\
\times \sum_{n=0}^{\infty} \frac{B_{\sigma}^{(\alpha,\beta;m)}(\mu - k + \frac{1}{2} + n, \mu + k + \frac{1}{2}) (\mu + \nu + \frac{5}{2})}{B(\mu + k + \frac{3}{2})} n! \\
\times P^{-\nu}_{\mu+k+\frac{3}{2}} \left( \frac{1}{\sqrt{4a^2 + 1}} \right) \left( \Re(\mu \pm k) > -\frac{1}{2}, \Re(\mu + \nu) > -\frac{5}{2} \right),
\]

where \( P_\nu^{\mu}(z) \) is the Legendre function of the first kind \([11, p. 34, Eq. (29)]\).

Proof. By using (18) and (14), expanding \( M_{\sigma,k,\mu}^{(\alpha,\beta;m)}(z) \) in terms of generalized extended beta function and changing the order of integration and summation, we get

\[
\int_0^\infty z M_{\sigma,k,\mu}^{(\alpha,\beta;m)}(z) J_\nu(az) \, dz = \sum_{n=0}^{\infty} \frac{B_{\sigma}^{(\alpha,\beta;m)}(\mu - k + \frac{1}{2} + n, \mu + k + \frac{1}{2}) (\mu + \nu + \frac{5}{2})}{B(\mu + k + \frac{3}{2})} n! \\
\times \int_0^\infty z^{\mu+n+\frac{3}{2}} e^{-\frac{z^2}{2}} J_\nu(az) \, dz.
\]

Using the known formula (see [3, p. 182, Entry (9)]):

\[
\int_0^\infty e^{-pt} t^\mu J_\nu(at) \, dt = \Gamma(\mu + \nu + 1) r^{-\mu-1} P_{\mu-\nu}^{\frac{p}{r}} \left( \frac{p}{r} \right) \\
\left( \Re(\mu + \nu) > -1, r = (p^2 + a^2)^{\frac{1}{2}} \right)
\]

in the above expression, after some simplification, we get the desired result. \( \square \)

4 Derivative of \( M_{\sigma,k,\mu}^{(\alpha,\beta;m)}(z) \)

For the generalized extended Whittaker function \( M_{\sigma,k,\mu}^{(\alpha,\beta;m)}(z) \), we have a differential formula given in Theorem 4.1.

Theorem 4.1. The following differential formula holds true:

\[
\frac{d^n}{dz^n} \left[ e^{\frac{z}{2}} z^{-\mu-\frac{1}{2}} M_{\sigma,k,\mu}^{(\alpha,\beta;m)}(z) \right] = \frac{(\mu - k + \frac{1}{2})_n}{(2\mu + 1)_n} e^{\frac{z}{2}} z^{-\mu-\frac{1}{2}} M_{\sigma,k,\mu}^{(\alpha,\beta;m)}(z),
\]

where \( n \in \mathbb{N} := \{1, 2, 3, \ldots\} \).
Proof. We recall the $n^{th}$ derivative of the generalized extended confluent hypergeometric function $\Phi_{\sigma}^{(\alpha,\beta;m)}$ (see [7, Theorem 3.5]):

$$\frac{d^n}{dz^n}[\Phi_{\sigma}^{(\alpha,\beta;m)}(b; c; z)] = \frac{(b)_n}{(c)_n} \Phi_{\sigma}^{(\alpha,\beta;m)}(b + n; c + n; z).$$  (34)

Now, we find from (18) that

$$\frac{d^n}{dz^n}[e^{\frac{1}{2} z - \mu - \frac{1}{2}} M_{\sigma,k,\mu}^{(\alpha,\beta;m)}(z)] = \frac{d^n}{dz^n}[\Phi_{\sigma}^{(\alpha,\beta;m)}(\mu - k + \frac{1}{2}; 2\mu + 1; z)].$$

By applying (34) in the above expression, in view of (18), we get the desired result.

## 5 Recurrence relations

Here we present some recurrence relations for the generalized extended Whittaker function $M_{\sigma,k,\mu}^{(\alpha,\beta;m)}(z)$ given in Theorem 5.1.

**Theorem 5.1.** Each of the following recurrence relations holds true:

$$\beta M_{\sigma,k,\mu}^{(\alpha,\beta-1;m)}(z) + (\alpha - \beta) M_{\sigma,k,\mu}^{(\alpha,\beta;m)}(z) - \sigma z^m B(\mu - m - k + \frac{1}{2}, \mu - m + k + \frac{1}{2}) M_{\sigma,k,\mu-m}^{(\alpha,\beta;m)}(z) = 0;$$  (35)

$$\beta(\beta - 1) M_{\sigma,k,\mu}^{(\alpha,\beta-1;m)}(z) - \beta(\beta - 1) M_{\sigma,k,\mu}^{(\alpha,\beta;m)}(z) + \frac{\sigma z^m B(\mu - m - k + \frac{1}{2}, \mu - m + k + \frac{1}{2})}{B(\mu - k + \frac{1}{2}, \mu + k + \frac{1}{2})} M_{\sigma,k,\mu-m}^{(\alpha,\beta;m)}(z) = 0;$$  (36)

$$(1 + \alpha - \beta) M_{\sigma,k,\mu}^{(\alpha,\beta+1;m)}(z) + (\beta - 1) M_{\sigma,k,\mu}^{(\alpha,\beta-1;m)}(z) = 0;$$  (37)

$$\beta M_{\sigma,k,\mu}^{(\alpha,\beta;m)}(z) - \beta M_{\sigma,k,\mu}^{(\alpha-1,\beta;m)}(z) + \frac{\sigma z^m B(\mu - m - k + \frac{1}{2}, \mu - m + k + \frac{1}{2})}{B(\mu - k + \frac{1}{2}, \mu + k + \frac{1}{2})} M_{\sigma,k,\mu-m}^{(\alpha,\beta+1;m)}(z) = 0;$$  (38)
We begin by the following recurrence relation of the confluent hypergeometric function $1F_1$

\[
(b - a) \, _1F_1(a - 1; b; z) + (2a - b) \, _1F_1(a; b; z) + z \, _1F_1(a; b; z) - a \, _1F_1(a + 1; b; z) = 0.
\]  

From the relation (41), we can derive the following integral formula:

\[
\begin{aligned}
& (\beta - \alpha) \frac{z^{\mu+\frac{1}{2}} \exp(-\frac{z}{2})}{B(\mu - k + \frac{1}{2}; \mu + k + \frac{1}{2})} \\
& \times \int_0^1 t^{\mu-k-\frac{1}{2}}(1-t)^{\mu+k-\frac{1}{2}} e^{zt} \, _1F_1 \left( \alpha - 1; \beta; -\frac{\sigma}{t^m(1-t)^m} \right) dt \\
& + \frac{(2\alpha - \beta) \, z^{\mu+\frac{1}{2}} \exp(-\frac{z}{2})}{B(\mu - k + \frac{1}{2}; \mu + k + \frac{1}{2})} \\
& \times \int_0^1 t^{\mu-k-\frac{1}{2}}(1-t)^{\mu+k-\frac{1}{2}} e^{zt} \, _1F_1 \left( \alpha; \beta; -\frac{\sigma}{t^m(1-t)^m} \right) dt \\
& - \frac{\sigma \, z^{\mu+\frac{1}{2}} \exp(-\frac{z}{2})}{B(\mu - k + \frac{1}{2}; \mu + k + \frac{1}{2})} \\
& \times \int_0^1 t^{\mu-m-k-\frac{1}{2}}(1-t)^{\mu+m+k-\frac{1}{2}} e^{zt} \, _1F_1 \left( \alpha; \beta; -\frac{\sigma}{t^m(1-t)^m} \right) dt \\
& - \frac{\alpha \, z^{\mu+\frac{1}{2}} \exp(-\frac{z}{2})}{B(\mu - k + \frac{1}{2}; \mu + k + \frac{1}{2})} \\
& \times \int_0^1 t^{\mu-k-\frac{1}{2}}(1-t)^{\mu+k-\frac{1}{2}} e^{zt} \, _1F_1 \left( \alpha + 1; \beta; -\frac{\sigma}{t^m(1-t)^m} \right) dt = 0.
\end{aligned}
\]
Applying the integral representation of $M^{(\alpha,\beta;m)}_{\sigma,k,\mu}(z)$ in (19) to the above expression is seen to yield the first recurrence relation (35).

A similar argument can establish the other formulas (36)-(40) by, respectively, using the following recurrence relations of $\mathbf{1}_F^1$ (see, e.g., [9, p. 19]):

\begin{equation}
\frac{b(b-1)}{1} \mathbf{1}_F^1(a; b-1; z) - \frac{b(b-1)}{1} \mathbf{1}_F^1(a; b; z) \\
- bz \mathbf{1}_F^1(a; b; z) + (b-a)z \mathbf{1}_F^1(a; b+1; z) = 0;
\end{equation}

\begin{equation}
(1+a-b) \mathbf{1}_F^1(a; b; z) - a \mathbf{1}_F^1(a+1; b; z) + (b-1) \mathbf{1}_F^1(a; b-1; z) = 0;
\end{equation}

\begin{equation}
b \mathbf{1}_F^1(a; b; z) - b \mathbf{1}_F^1(a-1; b; z) - z \mathbf{1}_F^1(a; b+1; z) = 0;
\end{equation}

\begin{equation}
ab \mathbf{1}_F^1(a; b; z) + bz \mathbf{1}_F^1(a; b; z) - (b-a)z \mathbf{1}_F^1(a; b+1; z) \\
- ab \mathbf{1}_F^1(a+1; b; z) = 0;
\end{equation}

\begin{equation}
(a-1) \mathbf{1}_F^1(a; b; z) + z \mathbf{1}_F^1(a; b; z) + (b-a) \mathbf{1}_F^1(a-1; b; z) \\
- (b-1) \mathbf{1}_F^1(a; b-1; z) = 0.
\end{equation}

\section{Concluding remarks}

Whittaker functions have diverse applications in the fields of mathematical physics including the spectral evolution resulting from the Compton scattering of radiation by hot electrons, modeling of the hydrogen atom, studies of the Coulomb Green’s function and so on. So those results in the present investigation may be (potentially) useful in some related areas of engineering sciences and mathematical physics.

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\section*{References}


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