On Cauchy-Euler Equation with a Bulge Function by Using Laplace Transform

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Abstract

In this paper, we study the nonhomogeneous second order differential equation of Cauchy-Euler equation with a bulge function. The Laplace transform, inverse Laplace transform and Taylor series expansion are used to derive the solution.

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1 Introduction

The Laplace transform can be employed in not only solving the linear ordinary differential equations with constant coefficient but also can be used with ordinary differential equations with variable coefficients. H. Kim [3] applied the Laplace transform to find the solution of a homogeneous Cauchy-Euler equation of the second order ODE. The research is to make an application to its difference equation and oscillation. M. S. Abualrub [4] found a special case of a non-homogenous linear Euler-Cauchy ODE. In this paper, we study the nonhomogeneous second order differential equation of Cauchy-Euler equation with a bulge function.
2 Preliminaries

We start off this study by giving out the the Laplace transform, inverse Laplace transform and the Taylor series expansion which can be used in this study.

**Definition 2.1.** The Laplace Transform [1]. Given a function \( f(t) \) defined for all \( t \geq 0 \), the Laplace transform of \( f \) is the function \( F \) defined as follow:

\[
F(s) = L\{f(t)\} = \int_{0}^{\infty} e^{-st} f(t) dt.
\]

(1)

for all values of \( s \) for which the improper integral converges

The Inverse transform [1]. If \( F(s) = Lf(t) \), then we call \( f(t) \) the inverse Laplace transform of \( F(s) \) and write

\[
y(t) = L^{-1}\{F(s)\}.
\]

(2)

The Cauchy-Euler equation. [2] An equation of the form

\[
a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_{n-2} x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + ... + a_1 x \frac{dy}{dx} + a_0 y = 0.
\]

(3)

is called the Cauchy-Euler equation. It is also known as Euler equation and as the equidimensional equation. We study the nonhomogeneous second order differential equation of Cauchy-Euler equation with a bulge function in the form \( t^2 y'' + aty' + by = e^{-\frac{(t-\ell)^2}{2}} \). The Laplace transform of the Cauchy-Euler equation of the first and second derivatives are expressed respectively by [3]

\[
L\{ty'\} = -F(s) - s \frac{d}{ds} F(s).
\]

(4)

and

\[
L\{t^2 y''\} = 2F(s) + 4s \frac{d}{ds} F(s) + s^2 \frac{d^2}{ds^2} F(s).
\]

(5)

3 The solution of Cauchy-Euler equation with a bulge function by using the Laplace transform

**Lemma 3.1.** The Laplace transform of the bulge function \( e^{-\frac{(t-\ell)^2}{2}} \) is expressed by [5, 6]

\[
L \left\{ e^{-\frac{(t-\ell)^2}{2}} \right\} = e^{-\frac{\ell^2}{2}} \left[ \frac{1}{s} + \frac{-1 + \ell^2}{s^3} + \frac{l(s^2 - 3 + l^2)}{s^4} \right].
\]

(6)
Proof. The Taylor series expansion $e^x$ is of the form

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \tag{7}$$

Therefore, by substituting equation (7) with $x = -\frac{(t-l)^2}{2}$, we obtain

$$e^{-\frac{(t-l)^2}{2}} = e^{-\frac{l^2}{2}t + e^{-\frac{l^2}{2}}lt + e^{-\frac{l^2}{2}}t^2 + \frac{l^3}{6}}.$$ \(\tag{8}
\)

By taking the Laplace transform to equation (8) and using the fact that the Laplace transform is linear, we derived

$$L\left\{e^{-\frac{(t-l)^2}{2}}\right\} = e^{-\frac{l^2}{2}} \left[ \frac{1}{s} + \frac{-1 + l^2}{s^3} + \frac{l(s^2 - 3 + l^2)}{s^4} \right]. \tag{9}\)

\[
\text{Lemma 3.2. The solution of the Cauchy-Euler equation with a bulge function}
\]

\[\begin{align*}
t^2 y'' + aty' + by &= e^{-\frac{(t-l)^2}{2}}, \quad y(0) = w, \quad y'(0) = q \tag{10}
\end{align*}\]

where $a$, $b$, $w$, $q$ are constants and $y(t)$ is unknown function can be expressed by

$$y(t) = \frac{1}{6} \left( \left( 6 \left( \Phi_1 \Phi_1 + \Phi_2 \Phi_2 \right) e^{\frac{t}{m}} + m \left( \Phi_3 + \Phi_4 (a^2 - 4b) \right) \right) e^{-\frac{l^2}{2}} 
+ 6b e^{\frac{t}{m}} \left( \Psi_1 (2mq - aw) + w(a^2 - 4b) \Phi_2 \right) / (b^4 m(a^2 - 4b)) \right).$$

where $\frac{d}{ds} = m$ and

$$\begin{align*}
\Psi_1 &= \sqrt{m^2(a^2 - 4b)} \sinh \left( \frac{t\sqrt{m^2(a^2 - 4b)}}{2m^2} \right), \\
\Psi_2 &= m(a^2 - 4b) \cosh \left( \frac{t\sqrt{m^2(a^2 - 4b)}}{2m^2} \right), \\
\Phi_1 &= 3ab^3 m^3 - 3ab^2 t^2 m^2 - 4bm^3 a^2 l^3 + ba^3 l^2 m^2 + lmb^2 a^2 + 12bmn^3 m^2 \\
&\quad + ab^3 - 6lm^3 b^2 + 2m^3 b^2 l^3 - 2lm b^3 - 3lm^3 a^4 + m^3 a^4 l^3 - ba^3 m^2, \\
\Phi_2 &= -b^3 + b^2 t^2 m^2 - b^2 m^2 + 2bm^3 a l^3 - 6balm^3 - alm b^2 - a^2 m^2 l^2 b \\
&\quad + a^2 m b - a^3 m^2 l + 3a^3 m^3 l, \\
\Phi_3 &= -24b^4 (lt + 1) + 6b^3 a^2 (lt + 1), \\
\Phi_4 &= 6a( -2b + a^2 ) (l^2 - 3)m^3 + 6(-1 + l^2 - 3lt + tl^3)(-b + a^2) bm^2 \\
&\quad + 6ab^2 (t^2 + l - t)m + l(l^2 - 3)tl^3 + 3t(l^2 b - b + l^3 am - 3lam)b^2.
\end{align*}$$
**Proof.** By taking the Laplace transform to equation (10), we have
\[ L\{t^2y''\} + aL\{ty'\} + bL\{y\} = L\left\{e^{-\frac{(t-1)^2}{2}}\right\}. \] (11)

Using the derivative property of Laplace transform to equation (11), it yields
\[ \frac{d^2}{ds^2}L\{y''\} - a\frac{d}{ds}L\{y'\} + bL\{y\} = L\left\{e^{-\frac{(t-1)^2}{2}}\right\}. \] (12)

By applying the Laplace transform of the second and first derivatives and lemma 3.1, we obtain
\[ \frac{d^2}{ds^2}\left[s^2F(s) - sy(0) - y'(0)\right] - a\frac{d}{ds}\left[sF(s) - y(0)\right] + bF(s) = e^{-\frac{l^2}{2}}\left[\frac{1}{s} + \frac{-1 + l^2}{s^3} + \frac{l(s^2 - 3 + l^2)}{s^4}\right]. \] (13)

where \( L\{y\} = F(s) \) and \( \frac{d}{ds} = m \). Then,
\[ m^2(s^2F(s) - sy(0) - y'(0)) - am(sF(s) - y(0)) + bF(s) = e^{-\frac{l^2}{2}}\left[\frac{1}{s} + \frac{-1 + l^2}{s^3} + \frac{l(s^2 - 3 + l^2)}{s^4}\right]. \] (14)

We can next solve for \( F(s) \) in equation (14), we have
\[ F(s) = (-m^2s^5y(0) - m^2s^4y'(0) + am^4s^2y(0) - e^{-\frac{l^2}{2}}s^3 + e^{-\frac{l^2}{2}}s - e^{-\frac{l^2}{2}}sl^2 - e^{-\frac{l^2}{2}}ls^2 + 3e^{-\frac{l^2}{2}}l - e^{-\frac{l^2}{2}}l^3) / (s^4(-m^2s^2 + am - b)). \] (15)

By imposing \( y(0) = w \), \( y'(0) = q \) to equation (15), we derive
\[ F(s) = (-m^2s^5w - m^2s^4q + am^4s^2w - e^{-\frac{l^2}{2}}s^3 + e^{-\frac{l^2}{2}}s - e^{-\frac{l^2}{2}}sl^2 - e^{-\frac{l^2}{2}}ls^2 + 3e^{-\frac{l^2}{2}}l - e^{-\frac{l^2}{2}}l^3) / (s^4(-m^2s^2 + am - b)). \] (16)

Then, the inverse Laplace transform can be used to equation (16) to obtain the solution of the Cauchy-Euler equation of the nonhomogeneous differential equation of equation (10) as
\[ y(t) = \frac{1}{6} \left(\left(6(\Psi_1\Phi_1 + \Phi_2\Psi_2) e^{\frac{ln}{6}} + m(\Phi_3 + \Phi_4(a^2 - 4b))\right) e^{-\frac{l^2}{2}} + 6b e^{\frac{ln}{6}} (\Psi_1(2mq - aw) + w(a^2 - 4b)\Psi_2)\right) / (b^4m(a^2 - 4b)). \]
4 Conclusion

In this paper, we study the nonhomogeneous second order differential equation of Cauchy-Euler equation with a bulge function which is denoted by

\[ f(t) = e^{-\frac{(t-l)^2}{2}} \]

where \( l \) is a positive constant. H. Kim [3] found the solution of Euler-Cauchy equation expressed by differential operator of homogeneous second order differential equation using Laplace transform. For this study, we found the way to solve the nonhomogeneous second order differential equation of Cauchy-Euler equation with a bulge function. We applied the Laplace transform, the inverse Laplace transform and the Taylor series expansion.

References


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