A Gauge Theory on Conformal Lie Groups

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Abstract

One studies a gauge theory of conformal Lie group C(d) over 3 and 4 dimensions. A gauge field with the gauge group C(d) doesn't present the second Chern class because of F ∧ F = 0 over 4-dimension, but the topological invariants are created by the first Chern class. Therefore one handles a gauge theory over 3-dimension.

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1 Introduction

According to Georgi and Glashow model, all fundamental interactions (without gravity) are severally representations of a unique group of unified fields [1]. For a unification model for the non-Abelian and Abelian gauge fields one can consider Glashow - Weinberg - Salam unification of weak and electromagnetic forces, \( U(1) \otimes SU(2) \equiv U(2) \) [2], [3], [4]. The weak interaction has massive gauge bosons which appear by a spontaneously symmetry breaking [5], [6] and the number of massive gauge fields is the dimension of subgroup of unbroken symmetry, (for example \( SU(2) \) for Weinberg - Salam model \( SU(2) \otimes U(1) \)), however, if there don’t exist unbroken part(s) of symmetry, then these fields are massless [7]. Therefore a group reduction mechanism given in [8] which the unified group \( SU(4) \otimes U(1) \) reduces to the Weinberg - Salam model \( SU(2) \otimes U(1) \) can be written as

\[
SU(4) \otimes U(1) \rightarrow SU(2) \otimes U(2) \rightarrow SU(2) \otimes SU(2) \otimes U(1). \tag{1}
\]
and this reduction mechanism in [8] present a quotient

$$\frac{SU(4) \otimes U(1)}{SU(2) \otimes SU(2) \otimes U(1)} \cong \mathbb{R}^9. \quad (2)$$

The number of the generators of the group $SU(4)$ having non vanishing diagonal elements is 3. Also, the number of the non-diagonal elements of an unitary Lie group $U(d)$ is $d^2 - d + d^2 - d$, where each $\frac{d^2 - d}{2}$ belongs to the hermitian conjugate. However we know that $u(4) \cong u(1) \otimes su(4)$ and $so(4) \cong su(2) \oplus su(2)$. Then we can mention from a different reduction mechanism follow

$$U(4) \rightarrow SU(4) \otimes U(1) \rightarrow SU(2) \otimes SU(2) \otimes U(1) \rightarrow SO(4) \otimes U(1). \quad (3)$$

This mechanism present also the same quotient like in eq. (2)

$$\frac{U(4)}{SO(4) \otimes U(1)} = \frac{SU(4) \otimes U(1)}{SO(4) \otimes U(1)} \cong \mathbb{R}^9. \quad (4)$$

Then the target point here is the product group $SO(4) \otimes U(1)$.

### 2 Geometry with Group $C(d)$

Let $X$ be any nondegenerate complex matrix of $d \times d$, that is $X \in GL(d, \mathbb{C})$. Then, using the definitions given for the real conformal Lie group and its Lie algebra by Kobayashi [9] we can write following definitions

$$C(d) = \{ X \in GL(d) \mid X^\top X = \lambda I \}, \quad (5)$$

$$\mathfrak{c}(d) = \{ X \in \mathfrak{gl}(d) \mid X^\top + X = -I \}, \quad (6)$$

where $\lambda = \sqrt{\det(X)} \in \mathbb{R} > 0$ is the conformal parameter and $I$ is identical matrix of $d \times d$. Therefore

$$\text{Tr}[\mathfrak{c}(d)] \neq 0. \quad (7)$$

Therefore, for an $\mathfrak{o}(d)$ - valued object, we can write a $\mathfrak{c}(d)$ - valued object follow

$$X = \begin{pmatrix} x_1^1 & x_1^2 & x_1^3 & \cdots \\ -x_1^2 & x_2^2 & x_2^3 & \cdots \\ -x_1^3 & -x_2^3 & x_3^3 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \in \mathfrak{c}(4), \quad \text{Tr}[X] \neq 0, \quad (8)$$

where $x_i^j$ are real valued.

Let $M$ be an Euclidean real 4 - manifold $M$ and $\nabla = d + A$ a connection on a principal $C(4)$ - bundle over the manifold $M$, where $A$ is the $\mathfrak{c}(4)$ - valued
connection 1-form. The curvature of this connection together with Bianchi identity is

\[ F = dA + A \wedge A, \]  
\[ d\nabla F = dF + A \wedge F - F \wedge A = 0, \]

and it consists also of the \( c(4) \)-valued 2-forms such that

\[ F = \begin{pmatrix} f_1^1 & f_1^2 & f_1^3 & \cdots \\ -f_2^1 & f_2^2 & f_2^3 & \cdots \\ -f_3^1 & -f_3^2 & f_3^3 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \in \Lambda^2(c(d)), \]

\[ \text{Tr}[F] \neq 0, \]  

where \( f_j^i = (f_j^i)_{\mu\nu} dx^\mu \wedge dx^\nu \in \Lambda^2(M) \) and \( A = a_j^i = (a_j^i)_{\mu} dx^\mu \in \Lambda^1(M) \). The connection 1-form must satisfy following boundary conditions

\[ \lim_{|x| \to \infty} A \sim g^{-1} dg, \text{ and } F = 0, \]
\[ \lim_{|x| \to 0} (f_j^i)_{\mu\nu} = f_0, \]

where \( g \in C(4) \) and \( f_0 \) a stable point point for the curvature (that is the gauge field strength). Therefore, we can choose the connection as follow

\[ A = \psi\omega = \psi g^{-1} dg \in \Lambda^1(c(4)), \]

where \( \psi \in C^\infty(M) \) and \( \omega \in \Lambda^1(c(4)) \) is the Maurer - Cartan 1-form of the group \( C(4) \) and it satisfies Maurer - Cartan equation

\[ d\omega + \omega \wedge \omega = 0. \]

Then the curvature becomes

\[ F = d\psi \wedge \omega + (\psi - \psi^2)d\omega. \]

Therefore the Bianchi identity gives

\[ \omega \wedge \omega \wedge \omega = 0. \]

Since \( \psi \in C^\infty(M) \), \( d\psi \wedge \omega = -\omega \wedge d\psi \), also because of \( \omega \wedge \omega \wedge \omega = 0 \) we get

\[ F \wedge F = 0. \]

This case appears as a problem because the wedge product of an ordinary 2-form by its own doesn’t vanish if \( \text{deg}(F \wedge F) \leq \text{dim}(M) \). Then, over 4-
dimension it cannot be \( F \wedge F \neq 0 \). On the other hand, the first and second Chern classes and the topological charge are defined such that

\[
\begin{align*}
\text{ch}_1 &= \frac{i}{2\pi} \text{Tr}[F], \\
\text{ch}_2 &= \frac{1}{2!} \left( \frac{i}{2\pi} \right)^2 (\text{Tr}[F] \wedge \text{Tr}[F] - \text{Tr}[F \wedge F]).
\end{align*}
\]

Therefore, for the curvature of the connection \( A = \psi \omega = \psi g^{-1} dg \) the Chern classes are obtained

\[
\text{ch}_1 = \frac{i}{2\pi} \text{Tr}[F], \quad \text{ch}_2 = 0.
\]

### 3 \( C(3) \) Gauge Theory Over 3-Manifolds

If one considers a 3-manifold of 4-manifold, then over this submanifold it must either \( F \wedge F = 0 \), so \( \deg(F \wedge F) \leq 3 \). This indicates the decomposition of the bundle of the 2-forms over 4-dimension into the subbundle of the 2-forms over 3-dimension:

\[
\Lambda^2(\mathbb{R}^4) \cong \Lambda^2(\mathbb{R}^3) \oplus \Lambda^2(\mathbb{R}^3)
\]

Let \( N \) be a 3-manifold as a submanifold of the 4-manifold \( M \) pointed out upper. The element 2-form \( f^i_j \) of the curvature matrix on the \( C(3) \) bundle is expressed with respect to the local coordinate \( \{ y^a \} \in \mathbb{R}^3 \) over the manifold \( N \) such that

\[
f^i_j = (f^i_j)_{ab} dy^a \wedge dy^b.
\]

Therefore, the action integral is written as

\[
S(A) = \int_N g \text{Tr}(f^i_j)^2 d\text{vol} < \infty,
\]

where

\[
d\text{vol} = dy^1 \wedge dy^2 \wedge dy^3, \quad \text{vol} = \int_N d\text{vol}.
\]

We have new boundary conditions as follow

\[
\begin{align*}
\lim_{|y| \to \infty} (a^i_j)_a &= (g^i_j)^{-1} \partial_a (g^i_j), \text{ and } (f^i_j)_{ab} = 0, \\
\lim_{|y| \to 0} (f^i_j)_{ab} &\sim f_0.
\end{align*}
\]

Under these boundary conditions we get a virtual action integral as follow

\[
I = \int_N g \text{Tr}(f^i_j - f_0 \delta^i_j)^2 d\text{vol},
\]

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where $\delta^i_j$ is Kronecker delta. The extrema of the variations with respect to each of the parameters $f_0$ and $g$ are

\[
\frac{\delta I}{\delta f_0} = 0 \iff \sum_{a<b} \text{Tr}(f^i_j)_{ab} = 3f_0 \quad (29)
\]
\[
\frac{\delta I}{\delta g} = 0 \iff \text{Tr}(f^i_j)^2 = 3(f_0)^2. \quad (30)
\]

Them are indeed the results of the self dual connection which will be given below. Thus we define a new topological charge concept as follow

\[
\Omega = gf_0 \int_N (\text{ch}_1 \wedge \Omega) \in \mathbb{Z}, \quad (31)
\]

where $\Omega$ is an auxiliary covariant constant 1 form and it can be choose such that

\[
\Omega = dy^1 - dy^2 + dy^3. \quad (32)
\]

Thus we find

\[
\Omega = g f_0 \int_N \text{Tr} \left( (f^i_j)_{12} + (f^i_j)_{13} + (f^i_j)_{23} \right) d\text{vol}
\]
\[
= 12g(f_0)^2 \times \text{vol}. \quad (33)
\]

The action integral given in eq. (24), together with eqs. (29) and (30) is rewritten as

\[
S(A) = E = 12(f_0)^2 \times \text{vol}. \quad (34)
\]

The action integral of the Lagrangian $\mathcal{L}$ is topologically bounded such that $\int_N \mathcal{L} \geq \Omega$, we write

\[
E \geq \Omega. \quad (35)
\]

4 Self $\Omega$ - Duality Over 3 - Dimension

The submanifold $N$ is a 3 - manifold. Therefore we can define a duality notion over 3 - dimension. Under a linear duality map

\[
*_{\Omega} : \Lambda^2(\mathbb{R}^3) \rightarrow \Lambda^2(\mathbb{R}^3) \quad (36)
\]

one can use a duality notion for the 2 - form on $\Lambda^2(\mathbb{R}^3)$ as follow

\[
*_{\Omega} F = *_{\Omega} \wedge * (F \wedge \Omega), \quad (37)
\]

where $\lambda$ is a free parameter. If

\[
F = \lambda *_{\Omega} F, \quad (38)
\]
then we call the self $\Omega$ - dual connection, where $\lambda = \text{Eigenvalues}[\Omega]$. One can choose the $\Omega$ such that

$$\Omega = dy^1 - dy^2 + dy^3.$$  \hspace{1cm} (39)

Then one gets

$$\lambda = \frac{1}{3}$$ \hspace{1cm} (40)

and the curvature 2- form $f$ becomes a self $\Omega$ - dual over 3 - dimension present. Thus we can easily write the matrix valued curvature 2 - form $(f^i_j)_{ab}$ of a self $\Omega$ - dual connection as follow

$$(f^i_j)_{ab}dy^a \wedge dy^b = (f^i_j)_{SD}(dy^{12} + dy^{13} + dy^{23}).$$ \hspace{1cm} (41)

This means that the each component of the curvature matrix is the same 2 - form.

$$\sum_{a<b}(f^i_j)_{ab} = 3(f^i_j)_{SD}, \quad \langle (f^i_j)_{SD} \rangle^2 = 3\langle (f^i_j)_{SD} \rangle^2.$$ \hspace{1cm} (42)

Because of eq. (30), we get

$$\langle (f^i_j)_{SD} \rangle^2 = \langle f_0 \rangle^2 = \text{Constant}.$$ \hspace{1cm} (43)

Then we rewrite

$$(f^i_j)_{ab}dy^a \wedge dy^b = f_0(dy^{12} + dy^{13} + dy^{23}),$$ \hspace{1cm} (44)

$$\sum_{a<b}(f^i_j)_{ab} = 3f_0, \quad \langle (f^i_j)_{SD} \rangle^2 = \langle f_0 \rangle^2.$$ \hspace{1cm} (45)

As known that the Hodge duality operator over 4 - dimension decomposes the 2 - forms into self (+) and anti self (−) dual parts which are orthogonal to each other, i.e $F = F_+ + F_-$ and $F_+ \wedge F_- = 0$. This decomposition presents a topological invariant term, called Pontrjagin index or instanton number:

$$k = \frac{1}{8\pi^2} \int_{\mathbb{R}^4} (\mathcal{F} \| F_+ \|^2 - \mathcal{F} \| F_- \|^2) \in \mathbb{Z}.$$ \hspace{1cm} (46)

For a self $\Omega$ - dual connection over 3 dimension, then we present another topological invariant like to the Pontrjagin index is defined as follow

$$k = \frac{1}{4\pi} \int_N \mathcal{F} \| F \wedge *_{\Omega} F \| \in \mathbb{Z}.$$ \hspace{1cm} (47)

Therefore, we have

$$k = \frac{12}{4\pi} \langle f_0 \rangle^2 \times \text{vol} \in \mathbb{Z},$$ \hspace{1cm} (48)

and so from the ratio $\frac{\Omega}{k}$

$$\frac{\Omega}{4\pi g} = k \in \mathbb{Z}.$$ \hspace{1cm} (49)

Therefore, the topological charge of a $C(3)$ gauge theory over 3 - dimension is proportional to the topological invariant term $k$. 
5 Conclusion and Discussion

In this text we studied a gauge theory $C(3)$ over 3-dimensional. The curvature of a connection $A = \psi g^{-1} dg$ over 4-dimensional doesn’t present the second Chern class because of $F \wedge F = 0$, however the topological concept is the first Chern class. Such a Chern class (21) is given for a $c(d)$-valued curvature 2-form (11). However the case $F \wedge F = 0$ is a problem because the wedge product of an ordinary 2-form by its own doesn’t vanish if $\text{deg}(F \wedge F) \leq \dim(M)$, and over 4-dimensional it cannot be $F \wedge F \neq 0$. Then the dimension of the base manifold must be less than 4, for example 3.

References


