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On the Number of Spanning Trees in Connected Cubic Circulants

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Abstract

Let $G_{2n}(a_1, a_2)$ be a connected cubic circulant of order $2n$ with two jumps a_1, a_2 such that $a_1 < a_2 = n$. We prove that the number of spanning trees, $t(G_{2n}(a_1, n))$, in $G_{2n}(a_1, n)$ is

$$t(G_{2n}(a_1, n)) = \frac{n}{2}[(2+\sqrt{3})^n + (2-\sqrt{3})^n - 2(-1)^{a_1}].$$

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1 Introduction

The number of spanning trees in undirected graph G, denoted by $t(G)$, is the total number of distinct connected spanning subgraphs of G that do not contain a cycle. A classic result of Kirchhoff [5], called matrix tree theorem, can be used to determine the number of spanning trees for any G. This is done through the evaluation of determinant of any cofactor of a special matrix associated with G. However, despite having efficient algorithms for evaluating such a determinant, the computations can be difficult and time consuming for the very large graphs. In addition, Kirchhoff's matrix tree theorem cannot easily support the analysis and comparison between the classes of graphs in terms of the number of spanning trees. Hence, there have been a lot of interest in recent years in deriving the explicit formulas for the special families of graphs. In particular, a few papers gave the explicit formulas for the subsets of well known graphs called circulants [1,7]. This is also the focus of this short note.

A *circulant graph* $G_n(a_1, a_2, ..., a_k)$ on n vertices with k pairwise distinct jumps $a_1, a_2, ..., a_k$ has vertices $i \pm a_1, i \pm a_2, ..., i \pm a_k \pmod{n}$ adjacent to each vertex i, where $a_j \leq n$ for $k \geq j \geq 1$. Without loss of generality, let $a_1 < a_2$ and let cubic circulant be denoted by $G_{2n}(a_1, a_2) = G_{2n}(a_1, n)$, where $a_1 < n$. In this work we give an explicit formula for the number of spanning trees in any connected $G_{2n}(a_1, n)$.

2 Main Result

Before presenting our main result we need the following four theorems. First, based on Boesch and Tindel [3] we have:

Theorem 2.1 [3] *Circulant graph $G_n(a_1, a_2, ..., a_k)$ is connected if and only if* $\gcd(n, a_1, a_2, ..., a_k) = 1$. □

Second, Sedlacek derived a formula for the number of spanning trees in a Möbius ladder [6]. The Möbius ladder, M_{2n}, is formed from cycle C_{2n} on $2n$ vertices labeled $v_1, v_2, ..., v_{2n}$ by adding edge $v_i v_{i+n}$ for every vertex v_i, where $i \leq n$.

Theorem 2.2 [6] The number of spanning trees in M_{2n} equals

$$t(M_{2n}) = \frac{n}{2}[(2+\sqrt{3})^n + (2-\sqrt{3})^n + 2] \quad \text{for } n \geq 2.$$

Third, another class of graphs for which we in [2] established an explicit formula has been derived based on a prism. Let the vertices of two disjoint and equal length cycles be labeled $v_1, v_2, ..., v_n$ in one cycle and $w_1, w_2, ..., w_n$ in the other. The prism, R_{2n}, is defined as the graph obtained by adding to these two cycles all edges of the form $v_i w_i$.

Theorem 2.3 [2] The number of spanning trees in R_{2n} equals

$$t(R_{2n}) = \frac{n}{2}[(2+\sqrt{3})^n + (2-\sqrt{3})^n - 2] \quad \text{for } n \geq 3.$$

Fourth, in [4] we recently established an isomorphism between the circulants and Cartesian products of 2 cycles of sizes n_1, n_2 (i.e., $C_{n_1} \Box C_{n_2}$) as follows:

Theorem 2.4 [4] *Circulant $G_n(a_1, a_2)$ is isomorphic to some Cartesian product of two cycles if and only if it is connected and* $\gcd(n, a_1)\gcd(n, a_2) = n$.

In particular, Theorem 2.4 allows cubic circulants and cubic Cartesian products of two cycles $C_{n_1}\square C_{n_2} = C_{n_1}\square C_2$ for $n_1 \geq 3$, where C_2 induces the cycles $v_i v_j v_i$ of length two that correspond to the edges (v_i, v_j) representing a 1-factor in $C_{n_1}\square C_2$.

We now present our main result. Without loss of generality we assume that $a_1 < n$ in $G_{2n}(a_1, a_2)$. The proof is based on showing that every connected circulant $G_{2n}(a_1, n)$ is either isomorphic to M_{2n} or R_{2n}.

Theorem 2.5 *If* $G_{2n}(a_1, n)$ *is a connected cubic circulant then*

$$t(G_{2n}(a_1, n)) = \frac{n}{2}[(2+\sqrt{3})^n + (2-\sqrt{3})^n - 2(-1)^{a_1}].$$

Proof. If a_1 is odd then $\gcd(a_1, 2n) = 1$ (otherwise $\gcd(a_1, n, 2n) \geq 2$ implying by Theorem 2.1 a disconnected $G_{2n}(a_1, n)$ – a contradiction), which by Theorem 2.1 means that there is a Hamilton cycle formed by a_1 in $G_{2n}(a_1)$. This implies that $G_{2n}(a_1, n)$ contains a Hamilton cycle formed by a_1. By edge symmetry of edges defined by a_1 there must be a simple path formed by edges a_1 of length n between vertices v_i, v_j that are adjacent based on jump $a_2 = n$. Hence $G_{2n}(a_1, n)$ is isomorphic to $G_{2n}(1, n)$, which by definition is isomorphic to M_{2n}. Consequently, by Theorem 2.2

$$t(G_{2n}(a_1, n)) = \frac{n}{2}[(2+\sqrt{3})^n + (2-\sqrt{3})^n + 2] \quad \text{for } a_1 \text{ odd}.$$

Consider now a_1 even. Suppose $\gcd(a_1, 2n) > 2$. Then $\gcd(a_1, n, 2n) \geq 2$ implying by Theorem 2.1 a disconnected $G_{2n}(a_1, n)$ – a contradiction. Hence, $\gcd(a_1, 2n) = 2$ and Theorem 2.4 implies that $G_{2n}(a_1, n)$ is isomorphic to a cubic $C_n\square C_2$ for $n \geq 3$, which by definition is isomorphic to R_{2n}. Consequently, by Theorem 2.3

$$t(G_{2n}(a_1, n)) = \frac{n}{2}[(2+\sqrt{3})^n + (2-\sqrt{3})^n - 2] \quad \text{for } a_1 \text{ even},$$

which concludes the proof. □

Based on proof of Theorem 2.5 we observe that for fixed n there are at most two distinct not isomorphic connected cubic circulants. Furthermore, the number of spanning trees is maximized for a_1 odd, and minimized for a_1 even if n is odd (otherwise, for n even all connected cubic circulants are isomorphic to one another).

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