Initial Boundary Value Problems for Viscoelastic Jeffreys Fluids

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Abstract

We study the initial boundary value problem for the nonlinear system, which describes the dynamics of an incompressible viscoelastic fluid with the Jeffreys constitutive law under the Navier slip boundary condition. We construct a global (in time) weak solution to this problem.

Mathematics Subject Classification: 35Q35, 35D30

Keywords: initial boundary value problem, weak solution, viscoelastic fluid, Jeffreys fluid, slip boundary condition

1 Introduction

The viscoelastic fluid of Jeffreys type is one of the classical non-Newtonian fluids with memory [7]. Mathematical modelling of such materials is an important task. The nonlinear equations of the Jeffreys model and other similar non-Newtonian models were studied by many authors (see e.g. [1], [4], [6], [8], [13], [15], [17], and the survey article [16]). In the majority of mathematical
works, the no-slip boundary condition is used. Nevertheless, many researchers emphasize the importance of studying the effects of slip at the solid surface (see e.g. [5], [14]). There are only a few contributions to slip problems for the Jeffreys model. The existence of weak steady solutions is proved in [2], [3]. Under some restrictions on the material constants and the data, Le Roux [9] showed the existence of a strong steady solution.

In this paper, we consider the initial boundary value problem for the non-linear system, which describes unsteady flows of an incompressible viscoelastic fluid of Jeffreys type in a bounded domain of $\mathbb{R}^n$ ($n = 2$ or $3$) under the Navier slip boundary condition. We give a weak formulation of this problem. Using the Faedo-Galerkin procedure and compactness arguments, we show the existence of a global (in time) weak solution.

2 Problem formulation and main results

Consider the motion and continuity equations for a homogeneous incompressible fluid:

$$\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = \text{Div} \mathbf{S} - \nabla p + \rho \mathbf{f}, \quad \text{div} \mathbf{v} = 0, \quad (2.1)$$

where $\rho$ is the density, $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ is the velocity of the fluid at a point $\mathbf{x} \in \mathbb{R}^n$ at time $t$, $p = p(\mathbf{x}, t)$ is the pressure, $\mathbf{f} = \mathbf{f}(\mathbf{x}, t)$ is the external force per unit mass, $\mathbf{S} = \mathbf{S}(\mathbf{x}, t)$ is the extra-stress. The precise form of the tensor $\mathbf{S}$ is given by a constitutive law, which depends on the fluid.

In this paper, we study the motion of a viscoelastic fluid with the Jeffreys constitutive law [7]:

$$\mathbf{S} + \lambda_1 \frac{d}{dt} \mathbf{S} = 2\eta \left( \mathbf{D}(\mathbf{v}) + \lambda_2 \frac{d}{dt} \mathbf{D}(\mathbf{v}) \right), \quad (2.2)$$

where $\mathbf{D}(\mathbf{v})$ is the strain velocity tensor, $\mathbf{D}(\mathbf{v}) = (\nabla \mathbf{v} + (\nabla \mathbf{v})^T)/2$, the symbol $d/dt$ denotes the material time derivative, $\lambda_1$ is the relaxation time, $\lambda_2$ is the retardation time, $0 < \lambda_2 < \lambda_1$, and $\eta$ is the viscosity coefficient, $\eta > 0$.

Relation (2.2) can be rewritten as

$$\mathbf{E} + \lambda_1 \left( \frac{\partial \mathbf{E}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{E} \right) = 2\alpha \eta \mathbf{D}(\mathbf{v}), \quad (2.3)$$

where $\mathbf{E}$ is the elastic part of $\mathbf{S}$,

$$\mathbf{E} = \mathbf{S} - 2\eta \lambda_2 \lambda_1^{-1} \mathbf{D}(\mathbf{v}), \quad (2.4)$$

and $\alpha = 1 - \lambda_2 \lambda_1^{-1}$. 
To write the motion equations and the constitutive law in dimensionless form, choose a characteristic length $l$ and a characteristic speed $V$ and introduce the following dimensionless variables and functions:

$$x^* = l^{-1}x, \quad t^* = l^{-1}Vt, \quad v^*(x^*, t^*) = V^{-1}v(x, t), \quad (2.5)$$

$$E^*(x^*, t^*) = l(\eta V)^{-1}E(x, t), \quad S^*(x^*, t^*) = l(\eta V)^{-1}S(x, t), \quad (2.6)$$

$$p^*(x^*, t^*) = l(\eta V)^{-1}p(x, t), \quad f^*(x^*, t^*) = \rho l^2(\eta V)^{-1}f(x, t). \quad (2.7)$$

Then, by writing the system of equations (2.1), (2.3), (2.4) in terms of dimensionless quantities (2.5)–(2.7) and omitting the asterisks, we get

$$\text{Re}\left(\frac{\partial v}{\partial t} + v \cdot \nabla v\right) + \nabla p - \text{Div} \ S = f, \quad \text{div} \ v = 0, \quad (2.8)$$

$$S = E + 2(1 - \alpha)D(v), \quad (2.9)$$

$$E + \text{We}\left(\frac{\partial E}{\partial t} + v \cdot \nabla E\right) = 2\alpha D(v), \quad (2.10)$$

where $\text{Re}$ is the Reynolds number, $\text{Re} = \rho l V \eta^{-1}$, $\text{We}$ is the Weissenberg number, $\text{We} = \lambda_1 V l^{-1}$.

Substituting (2.9) into (2.8), we obtain

$$\text{Re}\left(\frac{\partial v}{\partial t} + v \cdot \nabla v\right) + \nabla p - \text{Div} \ E - (1 - \alpha)\Delta v = f, \quad \text{div} \ v = 0. \quad (2.11)$$

Let us consider the system of equations (2.10), (2.11) in cylinder $\Omega \times (0, T)$, where $\Omega$ is a bounded locally Lipschitz domain in $\mathbb{R}^n$, $n = 2, 3$, with the boundary $\Gamma$. We assume that $\Gamma$ is impermeable. This yields that

$$v \cdot n = 0 \quad \text{on} \ \Gamma \times (0, T). \quad (2.12)$$

Moreover, we assume that the flow on the boundary $\Gamma$ is governed by the Navier slip condition (see [12] for the original reference):

$$kv = -[Sn]_\tau \quad \text{on} \ \Gamma \times (0, T).$$

Here $n = n(x)$ is the outer unit normal on $\Gamma$ at the point $x$, $v \cdot n$ is the scalar product of the vectors $v$ and $n$ in $\mathbb{R}^n$, $[.]_\tau$ denotes the tangential component of a vector, i.e., $[u]_\tau = u - (u \cdot n)n$, and $k = k(x)$ is the slip coefficient.

Taking into account (2.9), we rewrite the Navier slip condition as follows

$$kv = -[(E + 2(1 - \alpha)D(v))n]_\tau \quad \text{on} \ \Gamma \times (0, T). \quad (2.13)$$

In the sequel, we assume that the function $k$ satisfies the condition

$$0 < k_0 \leq k(x) \leq k_1 < \infty, \quad x \in \Gamma,$$
where \( k_0 \) and \( k_1 \) are constants.

Also, let us introduce the initial conditions:

\[
\begin{align*}
\mathbf{v}(x, 0) &= \mathbf{v}_0(x), & \mathbf{E}(x, 0) &= \mathbf{E}_0(x) \quad &\text{in } \Omega,
\end{align*}
\]

(2.14)

where \( \mathbf{v}_0 \) and \( \mathbf{E}_0 \) are given vector functions such that

\[
k\mathbf{v}_0 = - \left[ (\mathbf{E}_0 + 2(1 - \alpha) \mathbf{D}(\mathbf{v}_0)) \mathbf{n} \right]_\tau \quad \text{on } \Gamma.
\]

The purpose of this article is to prove the solvability of initial boundary value problem (2.10)–(2.14) in weak formulation.

First, we describe the concept of a weak solution to (2.10)–(2.14).

Let us introduce the following notation. For a given finite-dimensional space \( \mathbb{F} \), denote by \( L^p(\Omega, \mathbb{F}) \) and \( H^m(\Omega, \mathbb{F}) \) the Lebesgue and Sobolev spaces of functions \( w : \Omega \to \mathbb{F} \), respectively. The scalar product in \( L^2 \) will be denoted \( (\cdot, \cdot) \).

By definition, put \( X(\Omega, \mathbb{R}^n) = \{ \mathbf{v} \in H^1(\Omega, \mathbb{R}^n) : \text{div } \mathbf{v} = 0, \mathbf{v}|_{\Gamma} \cdot \mathbf{n} = 0 \} \).

Here the restriction of function \( \mathbf{v} \) to \( \Gamma \) is given by the formula \( \mathbf{v}|_{\Gamma} = \gamma_0 \mathbf{v} \), where \( \gamma_0 : H^1(\Omega, \mathbb{R}^n) \to H^{1/2}(\Gamma, \mathbb{R}^n) \) is the trace operator (see [11]).

Let us define the scalar product in \( X(\Omega, \mathbb{R}^n) \) by the following formula

\[
(v, w)_{X(\Omega, \mathbb{R}^n)} = 4\alpha(1 - \alpha)(D(v), D(w)) + 2\alpha \int_{\Gamma} k \mathbf{v} \cdot \mathbf{w} \, d\Gamma.
\]

Taking into account Korn’s inequality

\[
\|D(v)\|_{L^2(\Omega, \mathbb{R}^{n \times n})}^2 + \|\gamma_0 v\|_{L^2(\Gamma, \mathbb{R}^n)}^2 \geq C\|v\|_{H^1(\Omega, \mathbb{R}^n)}^2, \quad v \in H^1(\Omega, \mathbb{R}^n),
\]

we obtain that the norm \( \| \cdot \|_{X(\Omega, \mathbb{R}^n)} = (\cdot, \cdot)_{X(\Omega, \mathbb{R}^n)}^{1/2} \) is equivalent to the norm induced from the space \( H^1(\Omega, \mathbb{R}^n) \).

As usual, the symbols \( L^2(0, T; Z), \ C([0, T]; Z), \) etc., denote the Banach spaces of quadratically integrable, continuous, etc., functions on the segment \( [0, T] \) with values in some Banach space \( Z \). If \( Z \) is a function space (e.g. \( H^m(\Omega) \)), then we identify the elements of \( L^2(0, T; Z), \ C([0, T]; Z), \) etc., with functions defined on \( \Omega \times [0, T] \) by the formula

\[
\mathbf{v}(t)(x) = \mathbf{v}(x, t), \quad (x, t) \in \Omega \times [0, T].
\]

Finally, by \( \mathbb{R}^{n \times n}_s \) we denote the space of \( n \times n \) symmetric matrices.

Assume that

\[
\mathbf{v}_0 \in X(\Omega, \mathbb{R}^n), \quad \mathbf{E}_0 \in H^2(\Omega, \mathbb{R}^{n \times n}_s), \quad \mathbf{f} \in L^2(0, T; L^2(\Omega, \mathbb{R}^n)).
\]
Definition. A pair \((v, E) \in L_2(0, T; X(\Omega, \mathbb{R}^n)) \times L_\infty(0, T; L_2(\Omega, \mathbb{R}^{n \times n}))\) is called a weak solution of initial boundary value problem (2.10)–(2.14) if the equalities

\[
-\text{Re} \int_0^T (v, \varphi') \, dt - \text{Re}(v_0, \varphi(0)) - \text{Re} \sum_{i=1}^n \int_0^T (v_i v, \frac{\partial \varphi}{\partial x_i}) \, dt + \int_0^T (E, D(\varphi)) \, dt
+ 2(1 - \alpha) \int_0^T (D(v), D(\varphi)) \, dt + \int_0^T \int_\Gamma k v \cdot \varphi \, d\Gamma \, dt = \int_0^T (f, \varphi) \, dt,
\]

(2.15)

\[
\int_0^T (E, \Phi) \, dt - \text{We} \int_0^T (E', \Phi') \, dt - \text{We}(E_0, \Phi(0)) - \text{We} \sum_{i=1}^n \int_0^T (E, v_i \frac{\partial \Phi}{\partial x_i}) \, dt
= 2\alpha \int_0^T (D(v), \Phi) \, dt
\]

(2.16)

hold for all \(\varphi \in C^1([0, T]; X(\Omega, \mathbb{R}^n))\) and \(\Phi \in C^1([0, T]; H^2(\Omega, \mathbb{R}^{n \times n}))\) such that \(\varphi(T) = 0, \Phi(T) = 0\).

Remark. Identities (2.15) and (2.16) appear from the following reasoning. Suppose \((v, E, p)\) is a classical solution of problem (2.10)–(2.14). We can rewrite the first equality of (2.11) in the following form:

\[
\text{Re} \left( \frac{\partial v}{\partial t} + v \cdot \nabla v \right) + \nabla p - \text{Div}(E + 2(1 - \alpha)D(v)) = f.
\]

(2.17)

Consider a function \(\varphi \in C^1([0, T]; X(\Omega, \mathbb{R}^n))\) such that \(\varphi(T) = 0\). Multiplying both sides of equality (2.17) by \(\varphi\) and integrating over the domain \(\Omega \times (0, T)\), we obtain

\[
\text{Re} \int_0^T \left( \frac{\partial v}{\partial t}, \varphi \right) \, dt + \text{Re} \int_0^T \left( \sum_{i=1}^n v_i \frac{\partial v}{\partial x_i}, \varphi \right) \, dt + \int_0^T (\nabla p, \varphi) \, dt
- \int_0^T \left( \text{Div}(E + 2(1 - \alpha)D(v)), \varphi \right) \, dt = \int_0^T (f, \varphi) \, dt.
\]

(2.18)
Integrating by parts the terms on the left-hand side of (2.18), we have

\[
- \text{Re} \int_0^T (v, \varphi') \, dt - \text{Re}(v_0, \varphi(0)) - \text{Re} \int_0^T \sum_{i=1}^n \left( v_i v, \frac{\partial \varphi}{\partial x_i} \right) \, dt \\
+ \int_0^T \left( E + 2(1 - \alpha)D(v), D(\varphi) \right) \, dt - \int_0^T \int_{\Gamma} \left( (E + 2(1 - \alpha)D(v))n \right) \cdot \varphi \, d\Gamma \, dt \\
= \int_0^T (f, \varphi) \, dt. \quad (2.19)
\]

Further, taking into account (2.13), we obtain

\[
\int_0^T \int_{\Gamma} \left( (E + 2(1 - \alpha)D(v))n \right) \cdot \varphi \, d\Gamma \, dt = - \int_0^T \int_{\Gamma} k v \cdot \varphi \, d\Gamma \, dt. \quad (2.20)
\]

Combining (2.19) and (2.20), we obtain equality (2.15). Similarly, taking the \( L_2 \)-scalar product of (2.10) with a function \( \Phi \in C^1([0,T]; H^2(\Omega, \mathbb{R}^{n \times n})) \), \( \Phi(T) = 0 \), and integrating by parts, we obtain equality (2.16).

Now we formulate our main result.

**Theorem.** Initial boundary value problem (2.10)--(2.14) has at least one weak solution.

### 3 Proof of the Theorem

To prove the Theorem, we need the following lemma.

**Lemma 3.1** If \( H \) is a separable Hilbert space with the orthonormal basis \( \{ \varphi^j \}_{j=1}^\infty \), then the set \( \left\{ \sum_{j=1}^N h_j(t)\varphi^j : h_j \in C^1[0,T], N \in \mathbb{N} \right\} \) is dense in the space \( C^1([0,T], H) \).

This lemma can be proved by standard methods of analysis.

**Proof of the Theorem.** To construct a weak solution of (2.10)--(2.14), we apply the Faedo-Galerkin method.

Suppose \( \{ \chi^j \}_{j=1}^\infty \) is an orthonormal basis of \( X(\Omega, \mathbb{R}^n) \) and \( \{ Y^j \}_{j=1}^\infty \) is an orthonormal basis of \( H^2(\Omega, \mathbb{R}^{n \times n}) \). It can be assumed that the functions \( \chi^j \) and \( Y^j \) are smooth.
For each natural number $m$, we construct an approximate solution in the following form

$$v^m(x, t) = \sum_{j=1}^{m} \alpha_{mj}(t) \chi^j(x), \quad E^m(x, t) = \sum_{j=1}^{m} \beta_{mj}(t) Y^j(x),$$

where $\alpha_{mj}$ and $\beta_{mj}$ are unknown functions.

Consider the following auxiliary problem: For fixed $m \in \mathbb{N}$, find a pair $(v^m, E^m)$ such that

$$\text{Re} \left( \partial v^m / \partial t, \chi^j \right) + \text{Re} \sum_{i=1}^{n} \left( v_i^m \partial v^m / \partial x_i, \chi^j \right) + \left( E^m, D(\chi^j) \right) + 2(1 - \alpha)(D(v^m), D(\chi^j)) + \int_{\Gamma} k v^m \cdot \chi^j \, d\Gamma = (f, \chi^j), \quad j = 1, \ldots, m, \quad (3.1)$$

$$(E^m, Y^j) + \text{We} \left( \partial E^m / \partial t, Y^j \right) + \text{We} \sum_{i=1}^{n} \left( \partial E^m / \partial x_i, v_i^m Y^j \right) = 2\alpha (D(v^m), Y^j), \quad j = 1, \ldots, m, \quad (3.2)$$

$$v^m(\cdot, 0) = \sum_{j=1}^{m} (v^0, \chi^j)_x(\Omega, \mathbb{R}^n) \chi^j, \quad E^m(\cdot, 0) = \sum_{j=1}^{m} (E^0, Y^j)_{H^2(\Omega, \mathbb{R}^n)} Y^j. \quad (3.3)$$

Suppose that a pair $(v^m, E^m)$ satisfies (3.1)–(3.3). For $(v^m, E^m)$, we will obtain a priori estimates independent of $m$.

Multiplying (3.1) by $\alpha_{mj}(t)$, summing over $j$ from 1 to $m$, and using the equality

$$\sum_{i=1}^{n} \left( v_i^m \partial v^m / \partial x_i, v^m \right) = 0,$$

we deduce that

$$\text{Re} \left( \partial v^m / \partial t, v^m \right) + (E^m, D(v^m)) + 2(1 - \alpha)(D(v^m), D(v^m)) + \int_{\Gamma} k |v^m|^2 \, d\Gamma$$

$$= (f, v^m). \quad (3.4)$$

Now we multiply (3.2) by $\beta_{mj}(t)$ and add the results for $j = 1, \ldots, m$. Taking into account the relation

$$\sum_{i=1}^{n} \left( v_i^m \partial E^m / \partial x_i, E^m \right) = 0,$$
we get

\[(E^m, E^m) + \text{Re}((\partial E^m/\partial t, E^m)) - 2\alpha(D(v^m), E^m) = 0. \quad (3.5)\]

Further, we multiply equality (3.4) by 2\alpha. Summing the obtained equality and (3.5), we get

\[2\alpha \text{Re}(\partial v^m/\partial t, v^m) + 4\alpha(1 - \alpha)(D(v^m), D(v^m)) + 2\alpha \int_\Gamma k |v^m|^2 d\Gamma = 0. \quad (3.6)\]

Integrating (3.6) from 0 to \(t\), we have

\[2\alpha \text{Re}(v^m, v^m) + 4\alpha(1 - \alpha) \int_0^t (D(v^m), D(v^m)) ds + 2\alpha \int_0^t \int_\Gamma k |v^m|^2 d\Gamma ds + \int_0^t (E^m, E^m) ds + \text{We}(E^m, E^m) = 2\alpha \int_0^t (f, v^m) ds + 2\alpha \text{Re}(v^m(\cdot, 0), v^m(\cdot, 0)) + \text{We}(E^m(\cdot, 0), E^m(\cdot, 0)). \quad (3.7)\]

From equality (3.7), we obtain the following estimate

\[2\alpha \text{Re} \|v^m(\cdot, t)\|_{L^2(\Omega, \mathbb{R}^n)}^2 \leq \alpha \int_0^t \|f(\cdot, s)\|_{L^2(\Omega, \mathbb{R}^n)}^2 ds + \alpha \int_0^t \|v^m(\cdot, s)\|_{L^2(\Omega, \mathbb{R}^n)}^2 ds + 2\alpha \text{Re} C_0 \|v^0\|_{X(\Omega, \mathbb{R}^n)}^2 + \text{We} \|E^0\|_{H^2(\Omega, \mathbb{R}^{n \times n})}^2, \]

where \(C_0\) is a constant. Applying Gronwall’s inequality, we get

\[\sup_{t \in [0, T]} \|v^m(\cdot, t)\|_{L^2(\Omega, \mathbb{R}^n)}^2 \leq C(\alpha, \text{Re}, T) \left(\alpha \|f\|_{L^2(0, T; L^2(\Omega, \mathbb{R}^n))}^2 + 2\alpha \text{Re} C_0 \|v^0\|_{X(\Omega, \mathbb{R}^n)}^2 + \text{We} \|E^0\|_{H^2(\Omega, \mathbb{R}^{n \times n})}^2\right), \quad (3.8)\]

where \(C(\alpha, \text{Re}, T) = \exp(T/(2\text{Re}))/(2\alpha \text{Re}).\)

It follows from (3.7) and (3.8) that

\[\sup_{t \in [0, T]} \|E^m(\cdot, t)\|_{L^2(\Omega, \mathbb{R}^{n \times n})}^2 \leq \text{We}^{-1} \left(1 + \alpha TC(\alpha, \text{Re}, T)\right) \left(\alpha \|f\|_{L^2(0, T; L^2(\Omega, \mathbb{R}^n))}^2 + 2\alpha \text{Re} C_0 \|v^0\|_{X(\Omega, \mathbb{R}^n)}^2 + \text{We} \|E^0\|_{H^2(\Omega, \mathbb{R}^{n \times n})}^2\right), \quad (3.9)\]
\[ \|v^m\|_{L^2(0,T;X(\Omega,\mathbb{R}^n))}^2 \leq (1 + \alpha T C(\alpha, \Re, T)) \left( \alpha \|f\|_{L^2(0,T,L^2(\Omega,\mathbb{R}^n))}^2 ight. \\
+ 2\alpha \Re C_0 \|v^0\|_{L^2(\Omega,\mathbb{R}^n)}^2 + \text{We} \left\| E^0 \right\|_{H^2(\Omega,\mathbb{R}^{n \times n})}^2 \right). \] (3.10)

From the estimates (3.8) and (3.9) it follows that problem (3.1)–(3.3) has a solution defined on \([0,T]\).

Now we prove that the limit of sequence \(\{(v^m, E^m)\}_{m=1}^\infty\) is a weak solution of problem (2.10)–(2.14). Estimates (3.9) and (3.10) show the existence of a pair \((v^*, E^*)\) and a subsequence \(\{(v^{m_i}, E^{m_i})\}\) such that

\[ v^{m_i} \rightharpoonup v^* \text{ weakly in } L^2(0,T;X(\Omega,\mathbb{R}^n)), \] (3.11)

\[ E^{m_i} \rightharpoonup E^* \text{ weakly in } L^2(0,T;L^2(\Omega,\mathbb{R}^{n \times n})). \] (3.12)

In analogy with the case of the Navier-Stokes equations (see [10]), we can apply the compactness theorem and prove that

\[ v^m \rightharpoonup v^* \text{ strongly in } L^2(0,T;L^2(\Omega,\mathbb{R}^n)). \] (3.13)

Consider arbitrary functions \(\eta \in C^1[0,T]\) and \(\psi \in C^1[0,T]\) such that \(\eta(T) = 0, \psi(T) = 0\). We multiply (3.1) by \(\eta\), integrate with respect to \(t\) and integrate by parts. The result is

\[ -\Re \int_0^T (v^m, \chi^j) \eta' dt - \Re(v^m(\cdot,0), \chi^j) \eta(0) - \Re \sum_{i=1}^n \int_0^T (v^m_i v^m, \frac{\partial \chi^j}{\partial x_i}) \eta dt \]

\[ + \int_0^T (E^m, D(\chi^j)) \eta dt + 2(1-\alpha) \int_0^T (D(v^m), D(\chi^j)) \eta dt + \int_0^T \int_{\Gamma} k v^m \cdot \chi^j d\Gamma \eta dt \]

\[ = \int_0^T (f, \chi^j) \eta dt. \] (3.14)

Similarly, from equation (3.2), we get

\[ \int_0^T (E^m, Y^j) \psi dt - \text{We} \int_0^T (E^m, Y^j) \psi' dt - \text{We} (E^m(\cdot,0), Y^j) \psi(0) \]

\[ - \text{We} \sum_{i=1}^n \int_0^T (E^m_i, v^m_i \frac{\partial Y^j}{\partial x_i}) \psi dt = 2\alpha \int_0^T (D(v^m), Y^j) \psi dt \] (3.15)
Taking into account (3.11)–(3.13), we pass to the limit \( m \to \infty \) in equalities (3.14) and (3.15). The result is

\[
- \text{Re} \int_0^T (v^*, \chi^j) \eta' \, dt - \text{Re}(v_0, \chi^j) \eta(0) - \text{Re} \sum_{i=1}^n \int_0^T \left( v_i^* v^*, \frac{\partial \chi^j}{\partial x_i} \right) \eta \, dt
\]

\[
+ \int_0^T (E^*, D(\chi^j)) \eta \, dt + 2(1 - \alpha) \int_0^T (D(v^*), D(\chi^j)) \eta \, dt + \int_0^T \int_{\Gamma} k v^* \cdot \chi^j \, d\Gamma \eta \, dt
\]

\[
= \int_0^T (f, \chi^j) \eta \, dt.
\]

\[
\int_0^T (E^*, Y^j) \psi \, dt - \text{We} \int_0^T (E^*, Y^j) \psi' \, dt - \text{We}(E_0, Y^j) \psi(0)
\]

\[
- \text{We} \sum_{i=1}^n \int_0^T \left( E^*, v_i^* \frac{\partial Y^j}{\partial x_i} \right) \psi \, dt = 2\alpha \int_0^T (D(v^*), Y^j) \psi \, dt.
\]

Applying Lemma 3.1, we obtain that the pair \((v^*, E^*)\) is a weak solution of problem (2.10)–(2.14). The theorem is proved.

References


Received: September 3, 2015; Published: October 1, 2015