Asset Allocation Models in a Generalized Heston Framework Using a Gradient Like Vector Optimization Algorithm

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Abstract

This paper deals with the problem of determining “stable” portfolio allocations over a given time interval. Two portfolio optimization problems are formulated in the multi-asset Heston framework. The portfolio return is modeled by using a generalization of the Heston stochastic volatility model and explicit formulas for the conditional expected mean, the variance and the transition probability density function of the portfolio return process are deduced. These formulas are used to define the objective functions of the portfolio optimization problems. The optimal Pareto sets of these problems are explored to determine allocations which belong to the optimal Pareto sets of both optimization problems and that belong to these sets over a time interval. An experiment on real data is proposed to illustrate the implications of the “stable” asset allocation.

Keywords: Stochastic volatility model, Kolmogorov backward equation, maximum likelihood function, calibration procedure
1 Introduction

This paper deals with the problem of determining a long-term optimal portfolio when the returns of the risky assets are modeled with a stochastic volatility model. The main result is the formulation of two long-term portfolio optimization problems and their solution through the use of new computational methods. These optimization problems are formulated assuming that the returns of the risky assets in the portfolio are modeled with a generalization of the Heston stochastic volatility model (GH model for short) [20].

The GH model is a stochastic volatility model that describes the behavior of $n$ assets having stochastic variances prescribed by a mean reverting process. This model is obtained by the Heston stochastic volatility model adding to the stochastic differential equation satisfied by each asset price an extra term (see [20] for further details). A semi-explicit formula for the transition probability density function associated with the stochastic process is derived in [20] and this formula is used to calibrate the GH model parameters with the maximum likelihood approach.

Under the assumption that the asset returns are prescribed by the GH model in this paper we derive a semi explicit formula for the transition probability density function associated with the stochastic process that describes the portfolio return. This formula is expressed by a one dimensional integral of an explicitly known elementary integrand function and it can be evaluated using simple quadrature rules such as the midpoint quadrature rule. Moreover, using this formula we derive explicit formulas of the conditional expected values of the portfolio return and of its variance (see formulas (44) and (45)).

These elementary formulas of the GH model make it easy to formulate portfolio optimization problems and to solve them using specific computational methods. We study two portfolio optimization problems that are formulated as bi-criteria optimization problems. The first optimization problem minimizes the conditional variance and maximizes the conditional expected value of portfolio return. The second one minimizes the conditional variance and maximizes the probability that the portfolio return will not fall below a given threshold. The first optimization problem reduces to Markowitz’s approach [12] under suitable choices of the parameters that define the optimization problem itself.

These vector optimization problems are solved using a suitable path following algorithm that determines the whole optimal Pareto front of the problems. The optimal Pareto sets of these two optimization problems are explored in order to define a long term allocation procedure. This procedure selects, if any, points that are optimal for both problems and optimal for several future horizons in order to reduce the re-allocation cost. The portfolio is defined by using the risky asset log-returns instead of the asset prices. Roughly speaking, this corresponds to consider the geometric mean
of the asset prices instead of the arithmetic mean. This choice is not new (see [4], [16], [14], [7] and the references therein) and is motivated by the fact that the geometric mean is less affected by extreme values than the arithmetic mean and that it is useful as a measure of central tendency for some positively skewed distributions.

The portfolio optimization problems proposed in this paper are inspired by recent studies of Herzog et al. [8], Mwambi and Mwamba [14], Stuer et al. [22], Cremers et al. [3], Estrada [5], Gotoh and Takeda [7]. These studies show that the changes in the financial markets suggest the need to introduce new objectives in portfolio optimization in order to define efficient hedging strategies.

Specifically, the work of Mwambi and Mwamba proposes an investment strategy in the framework of the mean-variance portfolio selection based on the assumption that the asset prices are log-normally distributed. Under this assumption they formulate and solve portfolio optimization problems. In Estrada’s paper [5] a comparative analysis between the Growth Optimal Portfolio (GOP) and maximization of the Sharpe ratio is proposed. Elton and Gruber [4] study the practicality of the use of the geometric mean in the portfolio selection. Specifically, they investigate an algorithm to maximize the geometric mean with assumption log-normally distributed asset prices and they prove that the maximum geometric mean lies on the efficient frontier in the mean variance space. The work of Platen [17] shows that the optimal portfolio selection plays a fundamental role in financial market modeling and the recent studies are devoted to select portfolios that maximize investors’ expected utilities such as those proposed by Cremers et al [3], and Tim and Kritzman [23].

In addition, Gotoh and Takeda [7] illustrate a portfolio optimization model which minimizes the upper and lower bounds of loss probability of the underlying return distribution. This is done by reducing the portfolio optimization problem to two fractional programming models based on ratios which include the value at risk in the numerator and some norm of the portfolio in the denominator.

Finally, Kim et al. [10] show that building robust portfolios reduces the uncertainty in their performance. To do this they formulate portfolio optimization models based on a suitable modification of the Markowitz model. Our work is in line with this recent literature. In fact, we determine robust portfolios using a suitable analytical treatment of the stochastic volatility model which describes the underlying returns.

The paper is organized as follows. In Section 2 we derive the stochastic process that prescribes the dynamic of the portfolio when the asset prices are described by the generalization of the Heston model introduced in [20]. In Section 3 we derive a formula for the probability density function associated
with the stochastic variable that describes the portfolio dynamics. In Section 4 we formulate two asset allocation problems as bi-criteria optimization problems and we propose some experiments on real data to show how the asset allocation models work.

2 A stochastic model for the asset log-return portfolio dynamics

Let \( c_i, i=1,2,\ldots,n \), be real nonnegative constants and \( \zeta = (c_1, c_2, \ldots, c_n)^T \in \mathbb{R}^n \) such that \( c_1 + c_2 + \ldots + c_n = 1 \). We consider the following asset log-return portfolio:

\[
R_t = \sum_{i=1}^{n} c_i x_{i,t}, \quad t > 0,
\]

where \( x_{i,t}, t > 0 \), is the log-returns relative to the asset price \( S_{i,t}, i = 1, 2, \ldots, n \), that is

\[
x_{i,t} = \ln \frac{S_{i,t}}{S_{i,0}}.
\]

The asset dynamics are described by equations

\[
dS_{i,t} = S_{i,t} \mu_i dt + S_{i,t} \sqrt{v_{i,t}} dW_{0}^i + S_{i,t} a_i dQ_{0}^i, \quad (3)
\]

\[
dv_{i,t} = \chi_i(\theta_i - v_{i,t}) dt + \varepsilon_i \sqrt{v_{i,t}} dW_{1}^i, \quad (4)
\]

where \( \mu_i, a_i, \chi_i, \theta_i, \varepsilon_i, \) for \( i = 1, 2, \ldots, n \), are suitable real constants, satisfying the following conditions:

\[
a_i, \chi_i, \theta_i, \varepsilon_i > 0, \quad \tag{5}
\]

\[
2\chi_i \theta_i > 1, \quad \tag{6}
\]

and \( W_{i,t}^0, Q_{i,t}^0, W_{1,t}^1 \) are standard Wiener processes such that \( W_{i,0}^0 = W_{1,0}^1 = Q_{i,0}^0 = 0, i = 1, 2, \ldots, n \). We assume that the following relations hold:

\[
E(dW_{i,t}^0 dQ_{j,t}^0) = 0 \quad \forall \ i, j, \quad E(dW_{i,t}^1 dQ_{j,t}^0) = 0 \quad \forall \ i, j, \quad (7)
\]

\[
E(dW_{i,t}^0 dW_{j,t}^1) = 0 \quad \forall \ i \neq j, \quad E(dW_{i,t}^0 dW_{i,t}^1) = \rho_i dt, \quad (8)
\]

\[
E(dW_{i,t}^0 dW_{j,t}^0) = 0 \quad \forall \ i \neq j, \quad E(dW_{i,t}^1 dW_{j,t}^0) = dt, \quad (9)
\]

\[
E(dW_{i,t}^1 dW_{j,t}^1) = 0 \quad \forall \ i \neq j, \quad E(dW_{i,t}^1 dW_{i,t}^1) = dt, \quad (10)
\]

\[
E(dQ_{i,t}^0 dQ_{j,t}^0) = \rho_{i,j} dt \quad \forall \ i \neq j, \quad E(dQ_{i,t}^0 dQ_{i,t}^0) = dt, \quad (11)
\]

where \( dW_{i,t}^0, dQ_{i,t}^0, dW_{i,t}^1 \) are the stochastic differentials of the Wiener processes, \( E(\cdot) \) denotes the expected value of \( \cdot \) and \( \rho_i, \rho_{i,j} \in [-1, 1] \) are constants known
as correlation coefficients. It follows that
\[ dx_{i,t} = \left( \mu_i - \frac{1}{2} a_i^2 - \frac{1}{2} v_i \right) dt + \sqrt{v_i} dW^{0}_{i,t} + a_i dQ^{0}_{i,t}, \quad t > 0, \ i = 1, 2, \ldots, n, \] (12)
\[ dv_{i,t} = \chi_i(\theta_i - v_{i,t}) dt + \varepsilon_i \sqrt{\chi_i} dW^{1}_{i,t}, \quad t > 0, \ i = 1, 2, \ldots, n. \] (13)
with the initial conditions:
\[ x_{i,0} = \tilde{x}_{i,0} = 0, \quad i = 1, 2, \ldots, n, \] (14)
\[ v_{i,0} = \tilde{v}_{i,0}, \quad i = 1, 2, \ldots, n, \] (15)
where \( \tilde{x}_{i,0}, \tilde{v}_{i,0}, \ i = 1, 2, \ldots, n, \) are random variables that we assume to be concentrated in a point with probability one. Eqs. (1), (12) and (13) imply that the portfolio return \( R_t, \ t > 0, \) is the solution of the following system of stochastic differential equations:
\[ dR_t = \sum_{i=1}^{n} (c_i \mu_i - \frac{c_i}{2} a_i^2 - \frac{c_i}{2} v_i) dt + \sum_{i=1}^{n} c_i \sqrt{v_i} dW^{0}_{i,t} + \sum_{i=1}^{n} c_i a_i dQ^{0}_{i,t}, \quad t > 0, \] (16)
\[ dv_{i,t} = \chi_i(\theta_i - v_{i,t}) dt + \varepsilon_i \sqrt{\chi_i} dW^{1}_{i,t}, \quad t > 0, \ i = 1, 2, \ldots, n. \] (17)
The equations (16)–(17) must be equipped with the initial condition:
\[ R_0 = \tilde{R}_0 = \sum_{i=1}^{n} c_i \tilde{x}_{i,0}, \] (18)
\[ v_{i,0} = \tilde{v}_{i,0}, \quad i = 1, 2, \ldots, n, \] (19)
where \( \tilde{R}_0, \tilde{v}_{i,0}, \ i = 1, 2, \ldots, n, \) are random variables that we assume to be concentrated in a point with probability one. For simplicity, we identify the random variables \( \tilde{R}_0, \tilde{x}_{i,0}, \tilde{v}_{i,0}, \ i = 1, 2, \ldots, n, \) with the points where they are concentrated. Eqs. (1) and (2) imply \( \tilde{R}_0 = 0. \) Let \( p_f(R, v_1, \ldots, v_n, R', v_{i}', \ldots, v_{n}', t'), R, R' \in \mathbb{R}, \ t, t', v_i, v_{i}' \in \mathbb{R}^+, \ i = 1, 2, \ldots, n, \ t' > t, \) be the transition probability density function associated with the stochastic process implicitly defined by (16)–(17), that is, the probability density function of having \( R_{t'} = R', \ v_{i,t'} = v_{i}', \) given that \( R_t = R, v_{i,t} = v_{i}, \ i = 1, 2, \ldots, n, \) when \( t' > t. \) To derive a formula for \( p_f \) in analogy with Lipton [11] (pages 602–605), we consider the backward equation satisfied by \( p_f \) as a function of the “past” variables \( (R, v_1, \ldots, v_n, t) \in \mathbb{R} \times \mathbb{R}^{(n+1)+}, \) that is:
\[ -\frac{\partial p_f}{\partial t} = \frac{1}{2} \frac{\partial^2 p_f}{\partial R^2} \sum_{i=1}^{n} c_i^2 (v_i + a_i^2) + \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 p_f}{\partial v_i^2} \varepsilon_i^2 v_i 
+ \sum_{i=1}^{n} \frac{\partial^2 p_f}{\partial R \partial v_i} c_i \varepsilon_i v_i \rho_i + \frac{1}{2} \frac{\partial^2 p_f}{\partial x_i \partial x_j} \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \rho_{i,j} c_i c_j a_i a_j 
+ \sum_{i=1}^{n} \frac{\partial p_f}{\partial R_i} c_i \left( \mu_i - \frac{a_i^2}{2} - \frac{v_i}{2} \right) + \sum_{i=1}^{n} \frac{\partial p_f}{\partial v_i} \chi_i (\theta_i - v_i) \] (20)
with the final condition:

\[ p_f(R, v_1, \ldots, v_n, t', R', v'_1, \ldots, v'_n, t') = \delta(R - R') \prod_{j=1}^{n} \delta(v_j - v'_j) \]  

(21)

where, as mentioned above, \( \delta(\cdot) \) denotes Dirac's delta. As shown in the next section we have:

\[
p_f(R, v_1, \ldots, v_n, t, R', v'_1, \ldots, v'_n, t') =
\frac{1}{2\pi} \int_{\mathbb{R}} dk \ e^{ik[(R'-R)-\tau \sum_{j=1}^{n} c_j (\mu_j - \frac{a_j^2}{2})] - \frac{n \sum_{j=1}^{n} 2x_j \theta_j (\nu_j^R + \zeta_j^R) \tau}{\varepsilon_j}}.
\]

\[
e^{-\frac{k^2}{2} \varepsilon_j^2 R^2 D \varepsilon_j} \left\{ \prod_{j=1}^{n} \frac{4s_{\beta j}}{\varepsilon_j^2 \varepsilon_j^{R}} e^{-2\chi_j \theta_j / \varepsilon_j^2 \ln(s_{\beta j}^R / (2\xi_j^R))} e^{-2v_j((\zeta_j^R)^2 - (\nu_j^R)^2)} s_{\beta j}^R / (\varepsilon_j^2 s_{\beta j}^R) \right\}.
\]

(22)

where \( \tau = t' - t, \nu_j^R, \zeta_j^R \) for \( j = 1, 2, \ldots, n \) are given by:

\[
\nu_j^R = -\frac{1}{2} \left( \chi_j + i k c_j \varepsilon_j \theta_j \right)
\]

(23)

\[
\zeta_j^R = \frac{1}{2} \left( 4(\nu_j^R)^2 + \varepsilon_j^2 (k^2 c_j^2 - i k_j) \right)^{1/2},
\]

(24)

and the quantities \( s_{\beta j}^R, s_{j,\gamma}^R \) and \( \tilde{v}_j^R, M_j^R, j = 1, 2, \ldots, n \) are given by:

\[
s_{\beta j}^R = 1 - e^{-2\zeta_j^R \tau}, \quad s_{j,\beta}^R = (\zeta_j^R - \nu_j^R) + (\zeta_j^R + \nu_j^R) e^{-2\zeta_j^R \tau}, \quad \tau > 0,
\]

(25)

\[
\tilde{v}_j^R = \frac{4v_j(\zeta_j^R)^2 e^{-2\zeta_j^R \tau}}{(s_{\beta j}^R)^2}, \quad M_j^R = \frac{2s_{\beta j}^R}{\varepsilon_j^R s_{j,\beta}^R}, \quad \tau > 0.
\]

(26)

Integrating equation (22) with respect to the “future” variances \((v'_1, v'_2, \ldots, v'_n)\) we obtain:

\[
p_M(R, v_1, \ldots, v_n, t, R', t') =
\frac{1}{2\pi} \int_{\mathbb{R}} dk \ e^{ik[(R'-R)-(t'-t) \sum_{j=1}^{n} c_j (\mu_j - \frac{a_j^2}{2})] - (t'-t) \sum_{j=1}^{n} 2x_j \theta_j (\nu_j^R + \zeta_j^R)}.
\]

\[
e^{-(t'-t)\frac{k^2}{2} \varepsilon_j^2 R^2 D \varepsilon_j} \left\{ \prod_{i=1}^{n} e^{-2\chi_i \theta_i / \varepsilon_i^2 \ln(s_{i,\beta}^R / (2\xi_i^R))} e^{-2v_i((\zeta_i^R)^2 - (\nu_i^R)^2)} s_{i,\beta}^R / (\varepsilon_i^2 s_{i,\beta}^R) \right\}
\]

(27)

Note that the function \( p_M \) depends on the stochastic variances at time \( t \) (in the past). The quantities \( v_1, v_2, \ldots, v_n \) are not really observable in the market so that there are at least three ways of acting:
i) take an average value on \( \mathbb{R}^{n^+} \);
ii) assign them estimating their values from data using for example high-low volatility estimator, such as for example, \( \hat{\sigma}_{t}^{2} = (\ln H_t - \ln L_t)^2/(4 \ln 2) \) or \( \hat{\sigma}_{t}^{2} = 0.5(\ln H_t - \ln L_t)^2 - 0.39(\ln p_t - \ln p_{t-1})^2 \) where \( H_t, L_t \) and \( p_t, p_{t-1} \) are the highest, the lowest, the closing and the opening prices of the asset considered (see [18] and the references therein for further details);
iii) consider the initial variances as model parameters in the calibration procedure.

In this paper we choose the third approach.

### 3 Transition probability density function for the log-return portfolio

Let us derive the integral representation formula for \( p_{f,R} \) (see formula (22)). Letting \( \tau = t' - t \), we can introduce the function \( p_b \) defined as follows:

\[
p_b(\tau, R, v_1, \ldots, v_n, R', v'_1, \ldots, v'_n) = p_f(R, v_1, \ldots, v_n, t, R', v'_1, \ldots, v'_n, t'), \quad t' = t + \tau, \quad \tau > 0
\]

(28)

In fact, it is easy to see that the function \( p_f \) is only a function of \( \tau = t' - t \) instead of being a function of \( t \) and \( t' \) separately. Using the fact that \( \tau = t' - t \) and (28), equation (20) can be rewritten as an equation for \( p_b \), that is:

\[
\frac{\partial p_b}{\partial \tau} = \frac{1}{2} \frac{\partial^2 p_b}{\partial R^2} \sum_{i=1}^{n} c_i^2(v_i + a_i^2) + \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 p_b}{\partial v_i^2} \varepsilon_i^2 v_i
\]

\[
+ \sum_{i=1}^{n} \frac{\partial^2 p_b}{\partial R \partial v_i} c_i \varepsilon_i v_i \rho_i + \frac{1}{2} \frac{\partial^2 p_b}{\partial x_i \partial x_j} \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \rho_{i,j} c_i c_j a_i a_j
\]

\[
+ \sum_{i=1}^{n} \frac{\partial p_b}{\partial R_i} c_i \left( \mu_i - \frac{a_i^2}{2} - \frac{v_i}{2} \right) + \sum_{i=1}^{n} \frac{\partial p_b}{\partial v_i} \chi_i (\theta_i - v_i)
\]

(29)

with initial condition:

\[
p_b(0, R, v_1, \ldots, v_n, R', v'_1, \ldots, v'_n) = \delta(R - R') \prod_{i=1}^{n} \delta(v_i - v'_i)
\]

(30)

and appropriate boundary conditions. Now we consider the following representation formula for \( p_b \):

\[
p_b(\tau, R, v_1, \ldots, v_n, R', v'_1, \ldots, v'_n) =
\frac{1}{(2\pi)^{n+4}} \int_{\mathbb{R}} dk e^{ikR} \int_{\mathbb{R}} dl_1 e^{il_1 v_1} \ldots \int_{\mathbb{R}} dl_n e^{il_n v_n} \hat{f}(\tau, R, v_1, \ldots, v_n, k, l_1, \ldots, l_n)
\]

(31)
where \( \hat{f} \) is the Fourier transform with respect to the “future” variables \((R', v'_1, v'_2, \ldots, v'_n)\) and \(k, l_i, i = 1, 2, \ldots, n\), are the conjugate variables in the Fourier transform of \( R' \) and \( v'_i, i = 1, 2, \ldots, n\). Letting \( l'_i = \frac{2}{\varepsilon_i} l_i, i = 1, 2, \ldots, n\), from formula (31) we obtain:

\[
p_b(\tau, R, v_1, \ldots, v_n, R', v'_1, \ldots, v'_n) = \frac{1}{(2\pi)^{n+1}} \left( \frac{2}{\varepsilon_1^2} \right)^n.
\]

\[
\int_R dk e^{ikR'} \int_R dl_1 e^{\frac{2}{\varepsilon_1} d_1 v_1} \ldots \int_R dl_n e^{\frac{2}{\varepsilon_n} d_n v_n} \hat{f}(\tau, R, v_1, \ldots, v_n, k, l_1, \ldots, l_n)
\]

with initial condition:

\[
f(0, R, v_1, \ldots, v_n, k, l_1, \ldots, l_n) = e^{-\frac{1}{2k} R} \prod_{i=1}^{n} e^{-\frac{1}{\varepsilon_i} l_i v_i}.
\]
in order to satisfy (34), it is sufficient for the functions $A$ and $B_i$, $i = 1, 2, \ldots, n$ to satisfy the following ordinary differential equations:

\[
\frac{dA}{d\tau}(\tau, k, l_1, \ldots, l_n) = -\frac{k^2}{2} \varepsilon^T D\Gamma D\varepsilon - \nu k \sum_{i=1}^{n} c_i \left( \mu_i - \frac{a_i^2}{2} \right) - \sum_{i=1}^{n} B_i \chi_i \theta_i \tag{37}
\]

\[
-\frac{dB_i}{d\tau}(\tau, k, l_1, \ldots, l_n) = \frac{\varepsilon_i^2}{2} B_i^2 + (\nu c_i \varepsilon_i \rho_i + \chi_i) B_i - \frac{1}{2} (k^2 c_i^2 - ik c_i) \tag{38}
\]

with initial condition:

\[
A(0, k, l_1, \ldots, l_n) = 0, \quad B_i(0, k, l_1, \ldots, l_n) = l_i, \quad i = 1, \ldots, n. \tag{39}
\]

Note that the initial condition (39) has been obtained by imposing (35) on the function $f$ given by (36). Equation (38) is a Riccati equation that can be solved elementarily, then integrating (37) with respect to $\tau$ we obtain:

\[
A(\tau, k, l_1, \ldots, l_n) = -\frac{k^2}{2} \varepsilon^T D\Gamma D\varepsilon \tau - \nu k \tau \sum_{i=1}^{n} c_i \left( \mu_i - \frac{a_i^2}{2} \right) - \sum_{i=1}^{n} \frac{2 \chi_i \theta_i}{\varepsilon_i^2} \ln \left( \frac{(\nu_i^R + \zeta_i^R - il_i) e^{-2\zeta_i^R \tau} + (il_i - \nu_i^R + \zeta_i^R)}{2\zeta_i^R} \right) \tag{40}
\]

\[
B_i(\tau, k, l_i) = \frac{(\nu_i^R - \zeta_i^R)(\nu_i^R + \zeta_i^R - il_i) e^{-2\zeta_i^R \tau} + (\nu_i^R + \zeta_i^R)(il_i - \nu_i^R + \zeta_i^R)}{(\nu_i^R + \zeta_i^R - il_i) e^{-2\zeta_i^R \tau} + (il_i - \nu_i^R + \zeta_i^R)} \tag{41}
\]

where

\[
\nu_i^R = -\frac{1}{2} (\chi_i + \nu c_i \varepsilon_i \rho_i), \tag{42}
\]

\[
\zeta_i^R = \frac{1}{2} \left( 4(\nu_i^R)^2 + \varepsilon_i^2 (k^2 c_i^2 - ik_i) \right)^{1/2}. \tag{43}
\]

Substituting equations (40) and (41) into equation (32) and integrating with respect to the variables $l_i$, $i = 1, 2, \ldots, n$ (see, for example, Oberhettinger [15]), an elementary computation gives formula (22).
4 Asset allocation problems: an application on real data

In this section we compare two asset allocation problems which are formulated as bi-criteria optimization problems.

The two allocation problems are based on the assumption that the asset dynamics are described by the stochastic model (3), (4) and, as a consequence, the portfolio dynamics is given by (16) and (17). Furthermore, the asset allocation weights should be positive and no short-selling is allowed.

The first asset allocation problem, named $P_1$, is based on a mean-variance approach and the second one, named $P_2$, is based on the minimization of the variance and on the maximization of the probability that the value of the portfolio will not fall below a given threshold. Let us go into detail. Firstly, we derive formulas for the mean value and the variance of the portfolio conditioned to the observations made at a given time. Using formulas (1), (46), (47), (49) we obtain for $t' > t$:

$$
\hat{R}_{t,t'}(c) = E(R_{t'} | R_t = \tilde{R}, v_{i,t} = \tilde{v}_i, i = 1, 2, \ldots, n) = \sum_{i=1}^{n} c_i \hat{x}_{i,t,t'} \quad (44)
$$

$$
\hat{v}_{R,t,t'}(c) = E((R_{t'} - \hat{R}_{t,t'})^2 | R_t = \tilde{R}, v_{i,t} = \tilde{v}_i, i = 1, 2, \ldots, n) = 
\sum_{i=1}^{n} c_i^2 \hat{v}_{i,t,t'} + 2(t' - t) \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} c_i c_j \hat{\rho}_{i,j,t,t'} \quad (45)
$$

where $\tilde{R} = \sum_{i=1}^{n} c_i \tilde{x}_i$ while $\tilde{x}_i$ and $\tilde{v}_i$, $i = 1, 2, \ldots, n$, are the observation made at time $t$. The quantities $\hat{x}_{i,t,t'}$, $\hat{v}_{i,t,t'}$, and $\hat{\rho}_{i,j,t,t'}$, $i \neq j$, $i, j = 1, 2, \ldots, n$, $t' > t$ are the expected value and the variance of the variable $x_{i,t'}$ conditioned to the observation made at $t$ (see [20] for further details):

$$
\hat{x}_{i,t,t'} = E(x_{i,t'} | x_{i,t} = \tilde{x}_i, v_{i,t} = \tilde{v}_i) = 
\tilde{x}_i + (t' - t)(\mu_i - \theta_i - \frac{1}{2}a_i^2) + \frac{(\theta_i - \tilde{v}_i)}{2\chi_i} \left(1 - e^{-\chi_i(t'-t)}\right),
$$

(46)


\[ \hat{v}_{i,t,t'} = E\left((x_{i,t'} - \hat{x}_{i,t'})^2 \mid x_{i,t} = \bar{x}_i, \nu_{i,t} = \bar{\nu}_i\right) = (t' - t)a_i^2 + \frac{\theta_i\varepsilon_i}{8\chi_i^3}\left(1 - e^{-\chi_i(t'-t)}\right)^2 \]

\[ + \hat{v}_i \left((t' - t) - \frac{1 - e^{-\chi_i(t'-t)}}{\chi_i}\right) \left[\left(\rho_i - \frac{\varepsilon_i}{2\chi_i}\right) \left(\rho_i - \frac{3\varepsilon_i}{2\chi_i}\right) + (1 - \rho_i^2)\right] \]

\[ + \frac{\hat{v}_i}{\chi_i}\left(1 - e^{-\chi_i(t'-t)}\right) \left[\left(\rho_i - \frac{\varepsilon_i}{2\chi_i}\right)^2 + (1 - \rho_i^2)\right] \] (47)

\[ + \hat{v}_i \left((t' - t)\varepsilon_i e^{-\chi_i(t'-t)}\left(-\rho_i + \frac{\varepsilon_i}{2\chi_i}\right)\right) \] (48)

and finally:

\[ \hat{\rho}_{i,j,t,t'} = E\left((x_{i,t'} - \hat{x}_{i,t'})(x_{j,t'} - \hat{x}_{j,t'}) \mid x_{i,t} = \bar{x}_i, \nu_{i,t} = \bar{\nu}_i, x_{j,t} = \bar{x}_j, \nu_{j,t} = \bar{\nu}_j\right) = a_i a_j \rho_{i,j} (t' - t), \] (49)

Let us now formulate the two optimization problems that model the asset allocation problems \( P_1 \) and \( P_2 \):

Problem \( P_1 \):

\[ \min_{c \in \mathbb{R}^n} f_{P_1,t,t'}(c), \ t < t', \] (50)

subject to the constraints

\[ c_1 + c_2 + \ldots + c_n = 1, \ c_i \geq 0, \ i = 1, 2, \ldots, n, \] (51)

where \( f_{P_1,t,t'} = (f_{1,P_1,t,t'}, f_{2,P_1,t,t'}) \) is the vector function defined by:

\[ f_{1,P_1,t,t'}(c) = -\hat{R}_{t,t'}(c), \ t < t' \] (52)

\[ f_{2,P_1,t,t'}(c) = \hat{v}_{R,t,t'}(c), \ t < t'. \] (53)

Problem \( P_2 \):

\[ \min_{c \in \mathbb{R}^n} f_{P_2,t,t'}(c), \ 0 < t < t', \] (54)

subject to the constraints

\[ c_1 + c_2 + \ldots + c_n = 1, \ c_i \geq 0, \ i = 1, 2, \ldots, n, \] (55)

where \( f_{P_2,t,t'} = (f_{1,P_2,t,t'}, f_{2,P_2,t,t'}) \) is the vector function defined by:

\[ f_{1,P_2,t,t'}(c) = -\int_{R_i^2}^{+\infty} dR' P_{M,R}(R(c), \bar{v}_1, \ldots, \bar{v}_n, t, R', t'), \ t < t', \] (56)

\[ f_{2,P_2,t,t'}(c) = \hat{v}_{R,t,t'}(c), \ t < t', \] (57)
where $R^*$ is a positive number that is the threshold for portfolio value and $p_{M,R}$ is given by (27). Note that we have highlighted the dependence on $c$ of $R$ in formula (56).

Problems $P_1$ and $P_2$ depend on the observation made at time $t$ of the asset log-returns and on a future time $t'$ (the horizon), $t' > t$. Both the asset allocation strategies try to minimize the portfolio variance at time $t'$ in the future. One strategy (Problem $P_1$) tries to maximize the mean value of the portfolio at a time $t'$ in the future given the fact that at time $t$ the values of the asset log-returns in the portfolio are $\tilde{x}_i$, $i = 1, 2, \ldots, n$. The other strategy (Problem $P_2$) tries to maximize the probability that the value of the portfolio is greater than a fixed threshold $R^*$ at time $t'$ given the values $\tilde{x}_i$, $i = 1, 2, \ldots, n$ of the asset log-returns at time $t$, $t < t'$.

The proposed models, using a kind of tracking procedure, try to get the “best” allocation in a future time. That is, calibrating today, at time $t$, the stochastic model (16), (17), assigning a maximum time horizon $T > 0$ we can solve (today) Problem $P_1$ and Problem $P_2$ for several values of the time horizon $T_j \leq T$ for $j = 1, 2, \ldots, n_T$. This corresponds to choose $t' = T_j$, $j = 1, 2, \ldots, k$, in formulas (44), (45). We obtain the optimal Pareto fronts $F(T_j)$ and the corresponding optimal Pareto sets $A(T_j) \subset \mathbb{R}^n$ for any $j = 1, 2, \ldots, n_T$.

This knowledge of the Pareto set $A(T_j)$, for $j = 1, 2, \ldots, n_T$, can be used in the decisional process to achieve some goals. In the following we try to pursue the following three goals:

**Goal 1** choose an asset allocation that is as near as possible to a point of the optimal Pareto set for any $T_j$ ($j = 1, 2, \ldots, n_T$);

**Goal 2** choose an asset allocation such that for any $T_j$ ($j = 1, 2, \ldots, n_T$) guarantees a portfolio variance below a given target;

**Goal 3** choose an asset allocation such that for any $T_j$ ($j = 1, 2, \ldots, n_T$) guarantees a portfolio return above a given target.

Goals 2 and 3 are classical ones and have been studied in the literature (see, for example, [12], [23], [14], [8]).

In pursuing Goal 1 we have the advantage of reducing the costs implied by the asset reallocation. In fact, roughly speaking, if there exists an asset allocation that belongs to the optimal Pareto set $A(T_j)$ for any $j = 1, 2, \ldots, n_T$, the reallocation cost is zero on average. However, if we cannot find an asset allocation that is optimal for every $T_j$ we choose an asset allocation today that is optimal at least for the first horizon $T_1$ and whose sum of Euclidean distances from the other optimal sets $A(T_j)$ is the smallest.

Note that the approach proposed here is based on the assumption that the asset log-return dynamics is described by the stochastic model (12), (13). This
could be considered a restriction of the proposed asset allocation model. However, our model is a generalization of the well known Heston model [9] that has been shown to describe satisfactorily asset dynamics in several research papers (see, for example, [21], [6], [1]). Furthermore, the generalization of the Heston model used to describe the portfolio dynamics allows for a correlation between the assets without losing semi-explicit formulas for the transition probability density function associated with the stochastic process implicitly defined by the model itself.

The asset allocation procedure can be summed up in three main steps:

(a) Given the log-returns \( \tilde{x}_{i,t_j}, \) \( i = 1, 2, \ldots, n, \) observed at \( t_j, \) \( j = 1, 2, \ldots, h, \) \( t_j \in [0,t], \) with \( t \) the current date, estimate the parameters of the stochastic model;

(b) Given a predetermined value of the portfolio \( R^* \) and a time horizon \( T, \) \( T > t \) solve problem \( P_1 \) (i.e. maximize the conditional mean and minimize the conditional variance) and problem \( P_2 \) (i.e. maximize the probability of exceeding a predetermined value of the portfolio \( R^* \) and minimize the conditional variance) for several time values \( T_j, j = 1, 2, \ldots, n_T, \) \( t_j \in [t,T]; \)

(c) Select on the optimal Pareto sets \( A(T_j), t_j \in [t,T] \) of problem \( P_1 \) and \( P_2 \) the point \( \xi^* \) that satisfies a given goal (Goals 1-3).

The portfolio can be re-balanced with a given frequency (i.e.: daily, monthly, quarterly, semiannual or annually) applying steps (a)-(c) with a time horizon depending on the re-balancing frequency.

We apply this procedure on real data. We consider the historical series of daily data of four market indexes: DAX, FTSE 100 UK, S&P 500 and FTSE MIB observed in the period going from March 5, 2010 to April 5, 2011. We calibrate the model parameters using the daily data from March 5, 2010 to March 5, 2011 and we use the data of April 2011 to study the behavior of the proposed strategies for the asset allocation (i.e. Problem \( P_1 \) and \( P_2 \)).

Step (a) produces a vector \( \Theta^* \) containing the parameters, the correlation coefficients and the initial stochastic variances of the stochastic volatility model (12), (13). Once computed the vector \( \Theta^* \), it is possible to solve the two asset allocation problems \( P_1 \) and \( P_2 \). Problems \( P_1, P_2 \) are two linearly constrained multiobjective optimization problems that we solve using the algorithm developed in [13] and [19].

This computational method is an interior point method based on a suitable direction that plays the role of projected gradient-like direction for the vector objective function. More specifically, the algorithm is based on the numerical integration of a dynamical system defined by a vector field that is a linear combination of descent directions for the objective functions. The limit points of
the trajectories solutions of the dynamical system belong to the optimal Pareto set so that integrating the dynamical system starting from a sufficiently large number of initial points belonging to the feasible region we are able to reconstruct the whole optimal Pareto set and the corresponding Pareto set. For the sake of the simplicity for technical details on this algorithm we refer to the papers [13] and [19].

5 Some experiments on real data

In this section we perform some experiments described using real data. In particular we consider four market indexes: DAX, FTSE 100 UK, S&P 500 and FTSE MIB observed in the period going from March 5, 2010 to April 5, 2011. In particular we use daily data of the period March 5, 2010 March 5, 2011 to calibrate the model parameters with the procedure described in [20] and the data of April 2011 are used to study the behavior of the optimal portfolios.

The observed log-return values of the indexes considered in the numerical experiment are shown in Figure 1.

Now we solve problem $P_1$ and $P_2$ that model the asset allocation problems using the algorithm presented in [19]. We calibrate our portfolio using first time horizon $t' = T_1$ (two weeks in the future) and then time horizon $t' = T_2$ (one month in the future). Our calibration day is March 5, 2011. So doing our time horizons are about April 15, 2011 and April 30, 2011. We set the threshold $R^* = \log(1 + \Delta R^*)$, with $\Delta R^* = 5\%$.

Figure 2 shows the optimal Pareto fronts obtained solving Problem $P_1$ (a) and Problem $P_2$ (b).
Figure 2: Optimal Pareto front obtained solving Problem $P_1$ (a) and Problem $P_2$ (b) for two values of the time horizon. Blue line - squares $T_1$ (two weeks), green line - stars $T_2$ (one month).

Let us denote with $O_{P_1,T_j}$ and $O_{P_2,T_j}$, for $j = 1, 2$, the optimal Pareto sets obtained solving Problem $P_1$ and Problem $P_2$ respectively with time horizons $T_1 = \text{two weeks}$ and $T_2 = \text{one month}$.

Figure 3: Some points of the optimal Pareto set obtained solving Problem $P_1$ (a) and Problem $P_2$ (b) for two values of the time horizon. Green stars $T_1$ (two weeks), blue stars $T_2$ (one month).

Figures 3 (a), (b) and 4 (a), (b) show the first three components of the vectors $\xi_i = (c_1, c_2, c_3, c_4)$ belonging to the optimal sets $O_{P_i,T_j}$, $i, j = 1, 2$. Note that we represent $(c_1, c_2, c_3)^T$ since we have $c_4 = 1 - c_1 - c_2 - c_3$ and that the distribution of the points $\xi = (c_1, c_2, c_3)$ varies passing from $O_{P_i,T_1}$ to $O_{P_i,T_2}$, $i = 1, 2$, or passing from $O_{P_i,T_j}$ to $O_{P_i,T_j}$, $j = 1, 2$. In particular we can see that the sets $O_{P_1,T_1} \cap O_{P_1,T_2}$ and $O_{P_2,T_1} \cap O_{P_2,T_2}$, $O_{P_1,T_1} \cap O_{P_2,T_1}$ are non empty sets.

These optimal points are interesting for two reasons. The first one is that distributing the assets according with one of these points we are guaranteed
that two goals are satisfied (the maximization of the conditioned mean and of the probability that the portfolio value will not fall below a fixed threshold).

The second one is that passing from a time horizon of two week to a time horizon of a month the portfolio should be slightly reallocated and this fact implies that on average the reallocation costs are reduced.

Finally we show how the Pareto front changes when $\Delta R^*$ increases. Figure 5 (a) shows the behavior of the Pareto fronts when $\Delta R^* = 5\%$ and $\Delta R^* = 10\%$ when the time horizon is a month and Figure 5 (b) shows the optimal Pareto sets. We can see that increasing $\Delta R^*$ we get the same probability for higher values of the conditioned variance that is a reasonable behavior. However the intersection of the two optimal sets is not empty.
6 Conclusions

This paper presents two asset allocation problems where the dynamics of the assets are described by a generalization of the Heston stochastic volatility proposed in [20]. The aim of the paper is to determine asset allocations that are “stable” over a time interval. The term “stable” means that the allocation belongs to the optimal Pareto front for a given time horizon, thus reducing the re-allocation costs. To efficiently solve the allocation problems we derive an integral representation formula for the probability density function associated with the portfolio return variable which we use to formulate Problem $P_2$. The empirical analysis shows that these optimal and “stable” allocations exist. However, and that they deserve further investigation. However, it would be interesting to study easily verifiable conditions for these stable allocations.

References


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