The Fundamental Theorem of Asset Pricing in $L^1$-valued Stochastic Integrals

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Abstract

This paper is devoted to the study of the stochastic integration from the aspect of order completeness. It studies an $L^1$-valued stochastic integral and whether the related markets admit a pricing functional, which is the analog of the equivalent martingale measure.

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1 Introduction

The presence of heavy-tails in continuous time models, in order to fit the modelling requirements poses the question of more general versions of the two FTAPs, mostly relied on the geometry of these spaces. We recall that seminal references about FTAP are [1, 2]. The need for more general versions of FTAP- and especially of the First one- is actually the main reason for the consideration of an $L^1$-valued stochastic integral (with respect to some filtered probability space $(\Omega, \mathcal{F}, P, \mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]})$ in the present paper. We recall one of the main results in [3]: (Order 1st Fundamental Theorem of Asset Pricing) [3, Th.20] Let $E$ be a Banach lattice and $M$ be a sublattice of $E$. If $M$ admits a strictly positive projection, then every strictly positive and continuous
functional \( f : M \to \mathbb{R} \), admits a strictly positive, continuous extension on \( E \). Also, if \( E \) is a Banach lattice and \( M \) is a sublattice of \( E \) such that every strictly positive and continuous functional \( f : M \to \mathbb{R} \), admits a strictly positive, continuous extension on \( E \), then \( M \) admits a strictly positive projection. In this paper, we apply [3, Th.20] on certain stochastic finance models, relying on the properties of Riesz Spaces. We introduce an \( L^1 \)-valued stochastic integral and we prove that the well-known properties of the Riemann integral hold, thanks to the order-completeness of the \( L^p \)-spaces \( 1 \leq p < \infty \). We first remind of the notion of order completeness. A Riesz space (vector lattice) is order complete if every non-empty, order bounded from above subset of it has a supremum. If \( E \) is a band in an order complete vector lattice, then \( B \) is a projection band, namely \( E = B \oplus B^d \), (Riesz Thm. [4, Th.8.20]). The projection \( P_B : E \to B \) is strictly positive (for the notion of strictly positive projection, see [3, Def.7]), since it is positive and \( P_B(x) = 0, x \in E^+ \), implies that since \( x = x_1 + x_2 \), where \( x_1 \in B \) and \( x_2 \in B^d \), \( x_1 = 0, x_2 \wedge 0 = 0 \) and \( x_2 \geq 0 \), hence \( x_2 = 0 \) and finally \( x = 0 \). The same situation is valid for Kantorovich-Banach spaces (or else KB-spaces), in which \( E^{**} = E \oplus E^d \). Such examples of spaces are reflexive Banach lattices like \( L^p(\Omega, \mathcal{F}, \mathbb{P}) \), \( 1 < p < \infty \) and \( AL \)-spaces. For the notions regarding [3], the reader of this paper may see the Section of Notions and Definitions of that paper. About rest notions regarding ordered linear spaces and moreover, Riesz Spaces, the reader may see the Appendix at the end of this paper.

2 Processes of Stochastic Integrals in \( L^1 \)

We take \( L^1 \) as a case of a non-reflexive KB space, containing most of the heavy-tail distributed random variables in \( L^p(\Omega, \mathcal{F}, \mathbb{P}) \). According to the Hölder Inequality, the product of a variable \( y_t \) in \( L^p \) (or a variable which arises from taking suprema and infima on a stochastic process \( y \)) and a variable which is the increment of another process \( x_t \) in \( L^q \), (where \( \frac{1}{p} + \frac{1}{q} = 1, p, q \geq 1 \)), is a variable in \( L^1 \). We also consider a time horizon \([0, T]\) and a partition

\[ \mathcal{P} = \{t_0 = 0 < t_1 < t_2 < \ldots < t_{n-1} < t_n = T\}, \]

of it. We also consider a filtration \( \mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]} \)-under the usual sense and a pair \((y, x)\) of \( \mathbb{F} \)-adapted stochastic processes \( y, x : [0, T] \times \Omega \to \mathbb{R} \). Given a partition \( \mathcal{P} \),

\[ m_i = \inf\{y_t | t \in [t_i, t_{i+1}], i = 0, 1, \ldots, n - 1\}, \]

\[ M_i = \sup\{y_t | t \in [t_i, t_{i+1}], i = 0, 1, \ldots, n - 1\} \in L^p(\Omega, \mathcal{F}_t, \mathbb{P}), \]

thanks to the order completeness of \( L^p \) with respect to the usual partial ordering, which makes it a vector lattice. Hence we obtain the following
Definition 1. The lower stochastic integral of \( y \) with respect to \( x \) under \( \mathcal{P} \), is equal to

\[
L(y, x, \mathcal{P}) = \sum_{i=0}^{n-1} m_i(x_{t_{i+1}} - x_{t_i}) \in L^1(\Omega, \mathcal{F}_T, \mathbb{P}).
\]

Definition 2. The upper stochastic integral of \( y \) with respect to \( x \) under \( \mathcal{P} \), is equal to

\[
U(y, x, \mathcal{P}) = \sum_{i=0}^{n-1} M_i(x_{t_{i+1}} - x_{t_i}) \in L^1(\Omega, \mathcal{F}_T, \mathbb{P}).
\]

Again, thanks to the order completeness of \( L^p \) with respect to the usual partial ordering, which makes it a vector lattice, we may take the following:

Definition 3. The lower stochastic integral of \( y \) with respect to \( x \), is equal to

\[
L(y, x) = \sup_{\mathcal{P}} L(y, x, \mathcal{P}) \in L^1(\Omega, \mathcal{F}_T, \mathbb{P}),
\]

where \( \mathcal{P} \) is a partition of \([0, T]\).

Definition 4. The upper stochastic integral of \( y \) with respect to \( x \), is equal to

\[
U(y, x) = \inf_{\mathcal{P}} U(y, x, \mathcal{P}) \in L^1(\Omega, \mathcal{F}_T, \mathbb{P}),
\]

where \( \mathcal{P} \) is a partition of \([0, T]\).

We notice that

\[
L(y, x) \leq U(y, x).
\]

The next Criterion of Existence of the Stochastic Integral may be called Riemann-Riesz Criterion, both due to the similarity of it to the Riemann integrability criterion for real functions and due to the reference to Riesz Spaces.

Theorem 5. (Riemann-Riesz) The process \( y \) is integrable with respect to \( x \), namely

\[
U(y, x) = L(y, x),
\]

if for any sequence \((y_n)_{n \in \mathbb{N}} \subseteq L^1(\Omega, \mathcal{F}_T, \mathbb{P}), y_n \downarrow 0, \) a partition \( \mathcal{P}_n \) of \([0, T]\) exists, such that

\[
U(y, x, \mathcal{P}_n) - L(y, x, \mathcal{P}_n) \leq y_n.
\]

Proof: If \( U(y, x) = L(y, x) \), then \( U(y, x) + \frac{y_1}{2} > U(\cdot, y, x, \mathcal{P}_{n_1}) \) and \( L(y, x) - \frac{y_2}{2} < L(y, x, \mathcal{P}_{n_2}) \). For any \( n_0 \geq n_1, n_2 \), since \( y_n \downarrow 0 \),

\[
U(y, x, \mathcal{P}_n) - L(y, x, \mathcal{P}_n) < (U(y, x) - L(y, x)) + \left( \frac{y_{n_0}}{2} + \frac{y_{n_0}}{2} \right) = y_{n_0}.
\]
Theorem 6. (Linearity) If \( \int_0^T y_t \, dx_t, \int_0^T w_t \, dx_t \) exist in \( L^1 \), then \( \int_0^T (y_t + w_t) \, dx_t \) also exists in \( L^1 \). Furthermore, if \( \lambda \in \mathbb{R} \), then \( \int_0^T (\lambda \cdot y_t) \, dx_t = \lambda \cdot \int_0^T y_t \, dx_t \).

Proof: For the case of a partition \( P_n \) of \([0, T] \) with \( n + 1 \) points \( \{0 = t_0 < \ldots < t_{n-1} < t_n = T\} \), \( m_i(y) + m_i(w) \leq M_i(y+w) \leq M_i(y) + M_i(w) \), for any \( i = 1, 2, \ldots, n \). These order relations imply \( L(y, x, P_n) + L(w, x, P_n) \leq L(y + w, P_n) \leq U(y + w, P_n) \leq U(y, x, P_n) + U(w, x, P_n) < U(y, x, P_n) + U(w, x, P_n) + g_n \), where \( g_n \downarrow 0 \). Hence from the application of the Riemann-Riesz Criterion for \( y, w \), we take

\[
\int_0^T (y_t + w_t) \, dx_t = \int_0^T y_t \, dx_t + \int_0^T w_t \, dx_t.
\]

If \( \lambda \geq 0 \) then \( m_i(\lambda \cdot y) = \lambda \cdot m_i(y) \), and \( M_i(\lambda \cdot y) = \lambda \cdot M_i(y) \), for any \( i = 1, 2, \ldots, n \). These equalities imply \( \lambda \cdot L(y, x, P_n) = L(\lambda \cdot y, x, P_n) \leq U(\lambda \cdot y, x, P_n) = \lambda \cdot U(y, x, P_n) \leq \lambda \cdot L(y, x, P_n) + \lambda \cdot y_n, y_n \downarrow 0 \), since \( y \) is integrable with respect to \( x \), from the Riemann-Riesz Criterion. If \( \lambda < 0 \), \( \lambda \cdot L(y, x, P_n) + \lambda \cdot y_n < \lambda \cdot U(y, x, P_n) = L(\lambda \cdot y, x, P_n) \leq U(\lambda \cdot y, x, P_n) = \lambda \cdot L(y, x, P_n) \). Hence in any case, from Riemann-Riesz Criterion again,

\[
\lambda \int_0^T y_t \, dx_t = \int_0^T \lambda \cdot y_t \, dx_t.
\]

Theorem 7. (Additivity) If \( \int_0^T y_t \, dx_t, \int_T^U y_t \, dx_t \) exist in \( L^1(\Omega, \mathcal{F}_U, \mathbb{P}) \), then \( \int_0^U y_t \, dx_t \) also exists and \( \int_0^T y_t \, dx_t + \int_T^U y_t \, dx_t = \int_0^U y_t \, dx_t \).

Proof:

For the case of a partition \( P_{n_1} \) of \([0, T] \) consisted by \( n_1 + 1 \) points \( \{0 = t_1 < \ldots < t_{n_1-1} < t_{n_1} = T\} \),

\[
U(y, x, P_{n_1}) - L(y, x, P_{n_1}) < \frac{1}{2} d_{n_1},
\]

while for a partition of \([T, U] \) consisted by \( n_2 + 1 \) points \( \{T = t_1 < t_2 < \ldots < t_{n_2-1} < t_{n_2} = U\} \),

\[
U(y, x, P_{n_2}) - L(y, x, P_{n_2}) < \frac{1}{2} d_{n_2}.
\]

We also consider the partition \( P = P_{n_1} \cup P_{n_2} \) of \([0, U] \). For the partition \( P \), we get

\[
U(y, x, P_{n_1+n_2-1}) = U(y, x, P_{n_1}) + U(y, x, P_{n_2}),
\]

\[
L(y, x, P_{n_1+n_2-1}) = L(y, x, P_{n_2}) + L(y, x, P_{n_2}).
\]
Theorem 11. (Interpolation) If both \( y, x \) on \([0, T], [T, U]\) are restrictions on \( y, x \) on \([0, U]\), respectively. Hence,

\[ U(y, x, \mathcal{P}_{n_1+n_2-1}) - L(y, x, \mathcal{P}_{n_1+n_2-1}) < d_{n_1+n_2-1}, \]

since \( d_n \downarrow 0, d_n \in L^1(\Omega, \mathcal{F}_U, \mathbb{P}) \) for any \( n \in \mathbb{N} \).

Proposition 8. If \( \int_0^T y_t dx_t \) exist in \( L^1(\Omega, \mathcal{F}_T, \mathbb{P}) \), then \( \int_0^T |y_t| dx_t \) also exists.

Proof:
For the case of a partition \( \mathcal{P}_n \) of \([0, T]\) with \( n + 1 \) points \( 0 = t_1 < ... < t_{n-1} < t_n = T \), if \( s, u \in [t_i, t_{i+1}], i = 0, ..., n \), we have that \( |y_s| - |y_u| \leq |y_s - y_u| \leq M_i(y) - m_i(y) \). Hence, \( M_i(|y|) - m_i(|y|) \leq M_i(y) - m_i(y), i = 0, ..., n \).

Hence since by the Riemann -Riesz Criterion, the stochastic integral \( \int_0^T y_t dx_t \) exists, hence the integral \( \int_0^T |y_t| dx_t \) exists in \( L^1 \). This is true since we may select

\[ \|\mathcal{P}_n\| = \sup\{|x_{t_{i+1}} - x_t|, i = 0, 1, ..., n\} = d_n \in L^q(\Omega, \mathcal{F}_T, \mathbb{P}) \downarrow 0, \]

hence

\[ U(|y|, x, \mathcal{P}_n) - L(|y|, x, \mathcal{P}_n) \leq U(y, x, \mathcal{P}_n) - L(y, x, \mathcal{P}_n) \leq \]

\[ \sum_{i=0}^n (M_i(y) - m_i(y))d_n = a_n \downarrow 0, \]

where \( a_n \in L^1(\Omega, \mathcal{F}_T, \mathbb{P}) \).

Corollary 9. The space of the stochastic processes \( \{y_t\}_{t \in [0,T]} \) such that \( \int_0^T y_t dx_t \) exist in \( L^1(\Omega, \mathcal{F}_T, \mathbb{P}) \), is a sublattice of the space \( \mathbb{F}\)-adapted processes.

Proof:
Since \( \int_0^T (y_t + w_t) dx_t = \int_0^T y_t dx_t + \int_0^T w_t dx_t, \int_0^T \lambda \cdot y_t dx_t = \lambda \cdot \int_0^T y_t dx_t \), while \( \int_0^T |y_t| dx_t \) exists, too. Hence the stochastic integrals for \( y \wedge w = \frac{1}{2}(y + w + |y - w|), y \vee w = \frac{1}{2}(y + w - |y - w|) \) exist, with respect to \( x \) over \([0, T]\).

Definition 10. A selection of intermediate points from a partition \( \mathcal{P} \) of \([0, T] \) is a set of points \( E = \{\xi_1, \xi_2, ..., \xi_n\} \), such that \( t_i \leq \xi_i \leq t_{i+1}, i = 0, 1, 2, ..., n \). The Riemann -Riesz Sum for \( \mathcal{P} \) and \( E \) is the element of \( L^1(\Omega, \mathcal{F}_T, \mathbb{P}) \):

\[ S(y, x, \mathcal{P}, E) = \sum_{i=0}^n y_{\xi_i}(x_{t_{i+1}} - x_{t_i}). \]

Theorem 11. (Interpolation) If \( y \) is integrable with respect to \( x \), then this is equivalent to the following: For any sequence \( d_n \downarrow 0 \), where \( (d_n)_{n \in \mathbb{N}} \subseteq L^1 \), and any selection of intermediate points \( E_n \) of a partition \( \mathcal{P}_n \)

\[ |S(y, x, \mathcal{P}_n, E_n) - \int_0^T y_t dx_t| < d_n. \]
Hence,

\[ L(y, x, P_{n1}) \leq \int_0^T y_t \, dt \leq U(y, x, P_{n1}), \]

and

\[ L(y, x, P_{n1}) \leq S(y, x, P_{n1}, E_{n1}) \leq U(y, x, P_{n1}). \]

Hence,

\[ |S(y, x, P_{n1}, E) - \int_0^T y_t \, dt| \leq U(y, x, P_{n1}) - L(y, x, P_{n1}) < d_n. \]

For the inverse, we begin from \(|S(y, x, P_n, E_n) - \int_0^T y_t \, dt| < d_n\) and we are going to apply the Riemann -Riesz Criterion. We consider two selections of intermediate points \(\{\xi_1, \xi_2, ..., \xi_n\}, \{\eta_1, \eta_2, ..., \eta_n\}\) with respect to \(P_n\), such that \(y_{\xi_i} < m_i(y) + d_n, M_i(y) - d_n < y_{\eta_i}, i = 0, 1, 2, ..., n\). Then, we obtain

\[ U(y, x, P_n) - L(y, x, P_n) = \sum_{i=0}^{n} M_i(x_{t_{i+1}} - x_{t_i}) - \sum_{i=0}^{n} m_i(x_{t_{i+1}} - x_{t_i}) \leq \]

\[ \leq \sum_{i=0}^{n} y_{\eta_i}(x_{t_{i+1}} - x_{t_i}) + d_n \sum_{i=0}^{n} (x_{t_{i+1}} - x_{t_i}) - \sum_{i=0}^{n} y_{\eta_i}(x_{t_{i+1}} - x_{t_i}) + d_n \sum_{i=0}^{n} (x_{t_{i+1}} - x_{t_i}) = \]

\[ (\sum_{i=0}^{n} y_{\eta_i}(x_{t_{i+1}} - x_{t_i}) - I) + \theta_n + (I - \sum_{i=0}^{n} y_{\eta_i}(x_{t_{i+1}} - x_{t_i}) + \theta_n < 4\theta_n, \]

if \(I = \int_0^T y_t \, dt, \theta_n = d_n \sum_{i=0}^{n} (x_{t_{i+1}} - x_{t_i}) \downarrow 0\), while \(d_n \downarrow 0\) in \(L^1\), since \(\mathbb{E}(|x_{t_i} - x_{t_{i-1}}|)\) exist for \(i = 1, 2, ..., n - 1\). Then the proof is complete.

3 The FTAP for spaces of \(L^1\)- Stochastic Integrals

Since the classic conditional expectation \(\mathbb{E}(.|\mathcal{F}_t)\) is a strictly positive projection, [3, Th.20] is applied, in case where \(\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}\) denotes the classic notion
of filtration. For this application, we have to deduce whether the space of the marketed contingent claims

$$ C_T = \{ c_T \in L^1(\Omega, \mathcal{F}_T, \mathbb{P}) | c_T = y_0 + \int_0^T y_t dx_t \}, $$

is a sublattice of $L^1(\Omega, \mathcal{F}_T, \mathbb{P})$.

**Theorem 12.** $C_T$ is a sublattice of $L^1(\Omega, \mathcal{F}_T, \mathbb{P})$, hence again a space of the form $L^1(\Omega, \mathcal{A}, \mathbb{P})$, where $\mathcal{A}$ is a sub-$\sigma$-algebra of $\mathcal{F}_T$, if the constant random variable $1$ is replicated by some $(y_t)_{t \in [0,T]}$.

**Proof:** It is obvious that if the space of integrands $y = (y_t)_{t \in [0,T]}$ is a sublattice $R$ of the $\mathbb{F}$-adapted stochastic processes $y : [0,T] \times \Omega \to \mathbb{R}$ such that $y_t \in L^p(\Omega, \mathcal{F}_t, \mathbb{P})$, this is equivalent to $y_t \lor w_t \in R_t$, $y_t \land w_t \in R_t$, where $R_t$ is the $t$-projection of $R$. For a given partition $P = \{0 = t_1 < t_2 < ... < t_{n-1} < t_n = T\}$ of $[0,T]$,

$$ m_i(y \lor w) \leq m_i(y) \lor m_i(w) \leq M_i(y) \lor M_i(w) \leq M_i(y \lor w), $$

$$ m_i(y \land w) \leq m_i(y) \land m_i(w) \leq M_i(y) \land M_i(w) \leq M_i(y \land w) $$

hold. Since the stochastic integral of $y \lor w, y \land w$ with respect to $x = (x_t)_{t \in [0,T]}$ exists, by the above order relations, implies that $C_T + R_0$ (the range subspace of the stochastic integrals) is a sublattice, hence $C_T$ is a sublattice of $L^1(\Omega, \mathcal{F}_T, \mathbb{P})$. If $1$ is replicated, then by [4, Th.13.11], $C_T = L^1(\Omega, \mathcal{A}, \mathbb{P})$.

**Theorem 13.** If $C_T = L^1(\Omega, \mathcal{A}, \mathbb{P})$, then [3, Th.20] is applied on such a market.

**Proof:** This implies that for the conditional expectation with respect to $\mathcal{A}$ is a strictly positive projection $P : L^1(\Omega, \mathcal{F}_T, \mathbb{P}) \to L^1(\Omega, \mathcal{A}, \mathbb{P})$. For any strictly positive functional $f_0$ of $L^1_+(\Omega, \mathcal{A}, \mathbb{P})$, $f_0$ admits a strictly positive extension on $L^1(\Omega, \mathcal{F}_T, \mathbb{P})$, through $P^*(f) = g$. $g$ provides the Radon-Nikodym derivative of what should be called equivalent martingale measure for $C_T$.

**References**


Appendix

A partially ordered vector space $E$ is a vector lattice if for any $x, y \in E$, the supremum and the infimum of $\{x, y\}$ with respect to the partial ordering defined by $P$ exist in $E$. In this case $\sup\{x, y\}$ and $\inf\{x, y\}$ are denoted by $x \vee y$, $x \wedge y$ respectively. If so, $|x| = \sup\{x, -x\}$ is the absolute value of $x$ and if $E$ is also a normed space such that $\| |x| \| = \| x \|$ for any $x \in E$, then $E$ is called normed lattice. If a normed lattice is a Banach space, then it is called Banach lattice. A Banach lattice $E$ whose norm has the property $\| x + y \| = \| x \| + \| y \|$, $x, y \in E_+$ is called AL-space. A set $S$ in a vector lattice $E$ is called solid if $|y| \leq |x|$ and $x \in S$ implies $y \in S$. A solid vector subspace of a vector lattice is called ideal. An ideal $I$ is a sublattice of $E$, i.e. a subspace of $E$ such that $x \vee y \in I, x \wedge y \in I$ if $x, y \in I$ respectively. A net $\{x_a\}_{a \in A}$ in a vector lattice $E$ is order convergent to $x$ if there is a net $\{y_a\}_{a \in A}$ in $E$ with $y_a \downarrow 0$, such that $|x_a - x| \leq y_a$ for each $a \in A$. This convergence is denoted by $x_a \xrightarrow{o} x$. A set $D$ in $E$ is order closed if $\{x_a\}_{a \in A} \subseteq D$ and $x_a \xrightarrow{o} x$, implies $x \in D$. If $D$ is also an ideal, then $D$ is called band. A Banach lattice has order continuous norm, if for any net $\{x_a\}_{a \in A} \subseteq E$ with $x_a \downarrow 0$, $\| x_a \| \downarrow 0$ holds. A Banach lattice $E$ which is a band in its second dual (in the sense of norm topology) is called Kantorovich-Banach space. If $S$ is a subset of a vector lattice $E$, then its disjoint complement is the set $S^d = \{ x \in E : |x| \wedge |y| \text{ for any } y \in S \}$. If for a vector lattice $E$ a band $B$ satisfies the property $E = B \oplus B^d$, then $B$ is called projection band. For more details on the content of this Section, the reader may see [4, Ch.8, Ch.9].

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