Geometry of Cylindrical Curves over Plane Curves

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Abstract

In the paper we study differential geometric invariants of a class of space curves lying on right cylinders. We associate to a regular plane curve a unique space curve and define a correspondence between these curves such that the straight lines joining corresponding points form a right cylinder. We examine the relations between the invariants of the original plane curve and the invariants of the obtained space curve. The particular cases of space curves derived from the involute of the circle and the logarithmic spiral are also discussed.

Mathematics Subject Classification: 53A04; 65D17; 68U05

Keywords: plane curve, space curve, cylindrical surface, focal curve of a space curve

1 Introduction

The circular helix is a famous space curve which has many remarkable properties. This curve lies on a circular cylinder and its tangent vector at each point makes a constant angle with the cylinder axis. Both curvature and
torsion of the circular helix are nonzero constants. There is a generalization of a circular helix called a cylindrical helix. This space curve lying on a cylinder has a tangent vector which makes a constant angle with a fixed direction. Izumiya and Takeuch studied the cylindrical helices in [6]. They introduced in other their paper [7] a new class of so-called slant helices. Any slant helix is a regular space curve whose principle normal vector at each point makes a constant angle with a fixed direction. Izumiya and Takeuch also pointed out that the principle normal vector of the circular helix is perpendicular to its axis, i.e. any circular helix is a slant helix. There are many applications of circular helix in mathematics, physics and engineering (see for instance [2, p. 378]).

In this paper, we investigate another type of cylindrical curves obtained directly from plane curves. The first two coordinate functions of such a cylindrical curve coincide with the coordinate functions of a plane curve. The third coordinate function of the same space curve is either a curve parameter \( t \), or a suitable function of this parameter. The class of considered cylindrical curves contains the class of cylindrical helices. We examine the relations between differential-geometric invariants of a plane curve and differential-geometric invariants of a corresponding cylindrical curve. Moreover, we obtain a parametrization of the focal curves of these cylindrical curves.

The paper is organized as follows. The next section is devoted to the study of differential geometric invariants of cylindrical curves with respect to the Euclidean motions group and the group of direct similarities. An algorithm for constructing a focal curve of cylindrical curve is presented in Section 3. The considerations in Section 2 are extended to a more general construction of a cylindrical curve in the last section. In particular, the properties of a self-similar space curve are discussed.

2 Invariants of cylindrical curves over plane curves

One approach for obtaining new regular space curves is based on the use of parameterized plane curves. These new curves lie on the right cylinders over plane curves, and therefore we call them cylindrical curves over plane curves. Let \( \mathbf{O}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \) be a right-handed orthonormal coordinate system in the Euclidean space \( \mathbb{E}^3 \). We consider a circular helix \( \gamma = \gamma(t), \ t \in \mathbb{R} \) with parametric equations \( \gamma(t) = (a.\cos t, a.\sin t, b.t), \ a > 0, \ a, b = \text{const} \). The orthogonal projection of \( \gamma \) on the coordinate plane \( \mathbf{O}\mathbf{e}_1\mathbf{e}_2 \) is a circle centered at the origin with radius \( a > 0 \) and having parametric equations \( \alpha(t) = (a.\cos t, a.\sin t, 0) \). Thus, we can represent the helix \( \gamma \) with the vector parametric equation \( \gamma(t) = \alpha(t) + b.t.\mathbf{e}_3 \). In the similar way we may determine
a class of space curves.

**Definition 2.1.** Let $\alpha = \alpha(t)$, $t \in I \subseteq \mathbb{R}$ be a plane $C^3$ curve in the Euclidean plane $\mathbb{E}_2 \equiv O\mathbf{e}_1\mathbf{e}_2$ with a vector parametric equation $\alpha(t) = (x(t), y(t), 0)$ parameterized by an arbitrary parameter $t \in I$. Then the space curve $\gamma = \gamma(t)$, $t \in I$ with a parametric representation

$$\gamma(t) = (x(t), y(t), b(t)) = \alpha(t) + b(t)\mathbf{e}_3,$$  \hspace{1cm} \text{b = const}  \hspace{1cm} (1)$$

is called a cylindrical curve over the plane curve $\alpha(t)$.

The cylindrical (or generalized) helix is a regular space curve in $\mathbb{E}^3$ whose tangent vector makes a constant angle with a fixed direction. It is easy to show that any cylindrical (generalized) helix possesses a parametrization in the form (1). Let $\gamma = \gamma(s)$, $s \in I$ be a unit-speed generalized helix. Suppose that its tangents make a constant angle $\phi \neq \{0, \pi/2\}$ with a fixed unit vector $\mathbf{I} \neq \mathbf{0}$. Let $O\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ be a right-handed orthonormal coordinate system in the Euclidean space $\mathbb{E}^3$, where $\mathbf{e}_3 \equiv \mathbf{I}$. If $\gamma(s) = (x(s), y(s), z(s))$ is an arc-length parametrization of $\gamma$, then $\cos \phi = \langle \mathbf{t}, \mathbf{e}_3 \rangle = \frac{dz(s)}{ds}$, where $\mathbf{t} = t(s)$ is the unit tangent vector of $\gamma$ and $\langle \ldots \rangle$ denotes a dot product. Hence, $z(s) = \cos \phi.s$, provided that $z(0) = 0$. Then, we can determine the curve $\gamma$ by the vector parametric equation $\gamma(s) = \alpha(s) + \cos \phi.s.\mathbf{e}_3$, where $\alpha(s) = (x(s), y(s), 0)$ is a plane curve in the coordinate plane $O\mathbf{e}_1\mathbf{e}_2$.

It is natural to find the relation between the Euclidean curvature $\kappa = \kappa(t)$ and the Euclidean torsion $\tau = \tau(t)$ of the cylindrical curve $\gamma = \gamma(t)$, $t \in I$ and the signed curvature $K = K(t)$ of the corresponding plane curve $\alpha = \alpha(t)$, $t \in I$. Further in this section, we denote by $\ldots$ the differentiation with respect to an arbitrary parameter and by $\prime \prime$ the differentiation with respect to an arc-length parameter of the curve.

**Theorem 2.2.** Let $\gamma = \gamma(t)$, $t \in I$ be a cylindrical curve with equation (1), where $\alpha = \alpha(t)$, $t \in I$ is its corresponding plane curve. If $\kappa = \kappa(t)$ and $\tau = \tau(t)$ are the curvature and the torsion of $\gamma$, and $K = K(t)$ is the signed curvature of $\alpha$, then

$$\kappa(t) = \frac{\sqrt{A(t)}}{(\sqrt{s^2 + b^2})^3}, \hspace{1cm} \tau(t) = \frac{b.s. \left[ K.(3.\dot{s}^2 - \ddot{s}.s + K^2.\dot{s}^4) + \dot{K}.\dot{s}.s \right]}{A(t)}, \hspace{1cm} (2)$$

where $s = s(t)$ is the arc-length function of $\alpha$ and $A(t) = K^2.\dot{s}.(\dot{s}^2 + b^2) + b^2.\ddot{s}^2$.

**Proof.** Let $t = t(s)$ be the reverse function of the arc-length function $s = s(t)$ of $\alpha$. Replacing in (1) we have

$$\gamma(s) = \gamma(t(s)) = \alpha(t(s)) + b.t(s)\mathbf{e}_3 = \alpha(s) + b.t(s)\mathbf{e}_3. \hspace{1cm} (3)$$
We apply a differentiation with respect to the arc-length parameter $s$ of the plane curve $\alpha$. From (3) it follows that

$$\dot{\gamma} = \frac{d\gamma}{ds} = \alpha' + b.t'.\vec{e}_3 \quad \text{and} \quad \ddot{\gamma} = \frac{d^2\gamma}{ds^2} = \alpha'' + b.t''.\vec{e}_3.$$  \hspace{1cm} (4)

The structure equations of the curve $\alpha$ are

$$\alpha'' = K.J\alpha', \quad J\alpha'' = -K.\alpha',$$  \hspace{1cm} (5)

where $J$ is the complex structure of $\mathbb{R}^2$. It is well known that the Euclidean curvature $\kappa = \kappa(t)$ and the Euclidean torsion $\tau = \tau(t)$ of $\gamma$ with respect to an arbitrary parameter $t$ are given by

$$\kappa = \frac{\|\dot{\gamma} \times \ddot{\gamma}\|}{\|\dot{\gamma}\|^3} \quad \text{and} \quad \tau = \frac{\det(\dot{\gamma}, \ddot{\gamma}, \dddot{\gamma})}{\langle \dot{\gamma} \times \ddot{\gamma}, \dddot{\gamma} \rangle},$$  \hspace{1cm} (6)

where $\cdot \times \cdot$ is the vector cross product of $E^3$, $\langle \cdot, \cdot \rangle$ denote the scalar (or dot) product, and $\| \cdot \|$ is the length of the vector. Since $\alpha', J\alpha', \vec{e}_3$ form an right-handed orthonormal frame of $\gamma$ then from (4) and (5) we have

$$\dot{\gamma} \times \ddot{\gamma} = (\alpha' + b.t'.\vec{e}_3) \times (K.J\alpha' + b.t''.\vec{e}_3) = K.\vec{e}_3 - b.t''.J\alpha' - K.b.t'.\alpha' \quad \text{and} \quad \dddot{\gamma} = K'.J\alpha' - K^2.\alpha' + b.t'''.\vec{e}_3.$$

Hence,

$$\kappa(s) = \frac{\sqrt{K^2 + b^2.(K^2.t'^2 + t''^2)}}{(\sqrt{1 + b^2.t'^2})^3}, \quad \tau(s) = \frac{K.b.t''' - b.t''.K' + K^3.b.t'}{K^2 + b^2.(K^2.t'^2 + t''^2)}.$$  \hspace{1cm} (7)

Using the equalities $s = s(t), \ t' = \frac{1}{s}, \ t'' = -\frac{s'}{s^3}, \ t''' = \frac{3.s'^2 - \dot{s}.s'}{s^5}$ and (7) we get (2).

**Corollary 2.3.** Let $\alpha = \alpha(s), s \in I$ be a unit-speed plane curve in $\mathbb{E}^2$ and let $\gamma = \gamma(s) = \alpha(s) + b.s.\vec{e}_3, s \in I$ be the corresponding cylindrical curve of $c$. If $\kappa$ and $\tau$ are the curvature and the torsion of $\gamma$ and $K$ is the signed curvature of $\alpha$ then $\kappa = \frac{|K|}{1+b^2}, \quad \tau = \frac{bK}{1+b^2}, \quad b = \text{const}.$

The proof follows immediately from (2), replacing $t$ by $s$.

**Remark 2.4.** The condition $\frac{\tau}{\kappa} = \text{const}$ characterizes the generalized helixes. According the last corollary we may construct a generalized helix as a cylindrical curve over an arbitrary unit-speed plane curve.
Differential-geometric invariants of the regular curves in \( \mathbb{E}^n \) with respect to the group of direct similarities are functions called shape curvatures (see [4] and [5]). Essential role in this consideration plays the arc-length parameter \( \sigma \) of the spherical tangent indicatrix of the curve. We will call this parameter a spherical arc-length parameter. If the shape curvatures are known functions of the spherical arc-length parameter, then they determine a curve up to a direct similarity of the Euclidean space. Usually, the shape curvatures in the Euclidean space \( \mathbb{E}^3 \) are called a shape curvature and a shape torsion.

**Corollary 2.5.** Let \( \alpha = \alpha(\sigma), \sigma \in I \) be a plane curve in \( \mathbb{E}^2 \), parameterized by a spherical arc-length parameter \( \sigma \), and let \( \gamma = \gamma(\sigma) = \alpha(\sigma) + b.\sigma.\vec{e}_3, \sigma \in I \) be the corresponding cylindrical curve of \( \alpha \) If \( \tilde{\kappa} \) and \( \tilde{\tau} \) are the shape curvature and the shape torsion of \( \gamma \) and \( \tilde{K} \) is the shape curvature of \( \alpha \), then

\[
\tilde{\kappa}(\sigma) = -\frac{\tilde{K} \left[ -1 + b^2.K^2 \left( -2.\tilde{K}^2 + \tilde{K} \right) + b^4.K^4 \left( 1 + \tilde{K}^2 + \tilde{K} \right) \right]}{\left[ 1 + b^2.K^2 \left( 1 + \tilde{K}^2 \right) \right]^{3/2}} \\
\tilde{\tau}(\sigma) = \frac{b.|K|. \left( 1 + b^2.K^2 \right)^{3/2} \left( 1 + \tilde{K}^2 - \tilde{K} \right)}{\left[ 1 + b^2.K^2 \left( 1 + \tilde{K}^2 \right) \right]^{3/2}}, \tag{8}
\]

where \( K \) is the Euclidean signed curvature of \( c \).

**Proof.** From [4] we have \( d\sigma = K.ds \) and \( \tilde{K}(\sigma) = \frac{1}{K} \frac{dK}{d\sigma} \). Replacing the last equalities in the relations (2) (Theorem 2.2) we get

\[
\kappa(\sigma) = \frac{|K|}{(1 + b^2.K^2)^{3/2}} \sqrt{1 + b^2.K^2 \left( 1 + \tilde{K}^2 \right)}, \quad \tau(\sigma) = \frac{b.K^2 \left( 1 + \tilde{K}^2 - \tilde{K} \right)}{1 + b^2.K^2 \left( 1 + \tilde{K}^2 \right)}. \tag{9}
\]

Let \( s_{\gamma} \) and \( \sigma_{\gamma} \) be the arc-length parameter and the spherical arc-length parameter of the curve \( \gamma \), respectively. Hence, the tangent vector of \( \gamma \) is \( \dot{\gamma} = \frac{d\gamma}{d\sigma} = \frac{d\alpha}{d\sigma} + b.\vec{e}_3 = \frac{1}{K}.\alpha' + b.\vec{e}_3 \) and the first derivative of the arc-length function of \( \gamma \) is \( \dot{s}_{\gamma} = \| \dot{\gamma} \| = \frac{\sqrt{1 + b^2.K^2}}{|K|} \). Applying the formulas \( d\sigma_{\gamma} = \kappa ds_{\gamma} \), \( \tilde{\kappa}(\sigma) = -\frac{1}{\kappa} \frac{d\kappa}{d\sigma} \frac{d\sigma}{d\sigma_{\gamma}} = -\frac{1}{\kappa} \frac{d\kappa}{d\sigma} \frac{1}{\kappa} \frac{d\sigma}{d\sigma_{\gamma}} \), \( \tilde{\tau}(\sigma) = \frac{\tau(\sigma)}{\kappa(\sigma)} \) from [4] and using the relations (9) we obtain (8) after some rearranges and simplifications. \( \square \)
3 Focal curves of cylindrical curves

A focal curve and focal curvatures of smooth curve in \( m + 1\)-dimensional Euclidean space \( \mathbb{E}^{m+1} \) were introduced by Uribe-Vargas in [8]. Let \( \gamma : I \to \mathbb{E}^{m+1} \) be a smooth curve parameterized by its arc-length \( s \). Suppose that the Euclidean curvatures of this curve \( k_1, k_2, \ldots, k_m \) are nonzero, and the Frenet frame of \( \gamma \) is \( t, n_1, n_2, \ldots, n_m \). The centers of osculating spheres of the curve \( \gamma = \gamma(s) \) form a new curve \( C_\gamma : I \to \mathbb{E}^{m+1} \) given by

\[
C_\gamma(s) = \gamma(s) + c_1 n_1 + c_2 n_2 + \ldots + c_m n_m.
\]

The last curve is called a focal curve of \( \gamma \), and the coefficients \( c_1, c_2, \ldots, c_m \) are smooth functions called focal curvatures of \( \gamma \). Note that the focal curve \( C_\gamma \) is well-defined if and only if the curve \( \gamma \) does not lie on a hypersphere in \( \mathbb{E}^{m+1} \). According to Theorem 2 in [8], relations between Euclidean curvatures and focal curvatures of curve \( \gamma = \gamma(s) \) are given by

\[
k_i = \frac{c_1 c_1' + c_2 c_2' + \ldots + c_{i-1} c_{i-1}'}{c_{i-1} c_i} \quad i \geq 2.
\]

The first focal curvature \( c_1 \) never vanishes and it is given by \( c_1 = \frac{1}{k_1} \).

The shape curvature and the shape torsion of a regular curve in \( \mathbb{E}^3 \) have a natural generalization for a regular curve in \( \mathbb{E}^{n+1} \). In fact, there exist \( n \) smooth functions \( \tilde{k}_1, \tilde{k}_2, \ldots, \tilde{k}_n \) which determine the curve \( \gamma : I \to \mathbb{E}^{m+1} \) up to a direct similarity. These functions are called shape curvatures (see [4] or Ch.6 in [5]). Relations between focal curvatures and shape curvatures of \( \gamma \) can be written in the form

\[
\tilde{k}_i = c_i c_i' - \sum_{i' < i} c_i c_i', \quad i \geq 2.
\]

In the rest of the paper, we consider curves only in \( \mathbb{E}^2 \) and \( \mathbb{E}^3 \).

Let \( \gamma : I \to \mathbb{E}^3 \) be a smooth curve of class \( C^3 \) parameterized by an arbitrary parameter \( t \). Assume that the Euclidean curvatures of this curve are \( \kappa_1(t) = \varkappa(t) > 0 \) and \( \kappa_2(t) = \tau(t) \neq 0 \). In [3] the following formulas for the shape curvatures are proved

\[
\tilde{\kappa}_1 = \varkappa = 3 ||\dot{\gamma} \times \ddot{\gamma}||^2 \langle \dot{\gamma}, \dddot{\gamma} \rangle - ||\dot{\gamma}||^2 \langle \dddot{\gamma} \times \dot{\gamma}, \dot{\gamma} \times \dddot{\gamma} \rangle
\]

\[
\tilde{\kappa}_2 = \tau = \frac{||\dot{\gamma} \times \ddot{\gamma}||^3}{||\dot{\gamma} \times \dddot{\gamma}||^3}. \det(\dot{\gamma}, \dddot{\gamma}, \dddot{\gamma}).
\]

Applying the well known formula for the centers of the osculating spheres of an arbitrary parameterized curve in \( \mathbb{E}^3 \) (see for instance [1, p. 82]) we get the parametric representation of the focal curve of \( \gamma \)

\[
C_\gamma(t) = \gamma(t) + c_1 n_1 + c_2 n_2,
\]
where \( \mathbf{n}_1 \) is the unit principal normal vector of \( \gamma \), \( \mathbf{n}_2 \) is the unit binormal vector of \( \gamma \), \( c_1 = \frac{1}{\kappa_1} \) and \( c_2 = \frac{\tilde{\kappa}_1}{\kappa_2} \).

The cylindrical curves considered in the previous section belong to the class \( C^3 \). Therefore, we can associate a focal curve (11) to any cylindrical curve over a plane curve of class \( C^3 \). The next algorithm shows how to obtain such an associated curve.

**Algorithm 3.1.** A construction of a focal curve for a given cylindrical curve.

1. Calculate the derivatives \( \dot{\gamma} = \frac{d}{dt} \gamma(t) \), \( \ddot{\gamma} = \frac{d^2}{dt^2} \gamma(t) \) and \( \dot{\gamma} = \frac{d^3}{dt^3} \gamma(t) \).

2. If \( \dot{\gamma} \times \ddot{\gamma} \neq (0,0,0) \) for any \( t \), then find the unit binormal vector \( \mathbf{n}_2 \), the unit principle normal vector \( \mathbf{n}_1 \) and go to Step 3, else go to Step 6.

3. Compute the curvature \( \kappa_1 \) and the torsion \( \kappa_2 \) of \( \gamma \).

4. If \( \kappa_2 \neq 0 \), then compute by (10) the shape curvature \( \tilde{\kappa}_1 \) and go to Step 5, else go to Step 6.

5. The parametrization of the focal curve of \( \gamma \) is

\[
C_\gamma(t) = \gamma(t) + \frac{1}{\kappa_1} \mathbf{n}_1 + \frac{\tilde{\kappa}_1}{\kappa_2} \mathbf{n}_2, \tag{12}
\]

6. The focal curve does not exist.

Now we consider an illustrative example of a focal curve of a cylindrical curve over an important plane curve.

**Example:** Consider an involute of a circle with parametrization

\[
\alpha(t) = (a \cos t + t \sin t, a \sin t - t \cos t), \quad a = \text{const} > 0, \quad t \in \mathbb{R}.
\]

The cylindrical curve \( \gamma(t) \) over \( \alpha(t) \) is given by

\[
\gamma(t) = (a \cos t + t \sin t, a \sin t - t \cos t, bt), \quad a > 0, \quad b > 0, \quad t \in \mathbb{R}.
\]

Since the vector \( \dot{\gamma} \times \ddot{\gamma} = (-ab(t \cos t + \sin t), -ab(t \sin t - \cos t), a^2 t^2) \) is nonzero for any \( t \), the space curve \( \gamma(t) \) is regular. For the unit principle normal vector field \( \mathbf{n}_1 \) and the unit binormal vector field \( \mathbf{n}_2 \) of \( \gamma \) we get

\[
\mathbf{n}_1 = \left( \frac{b^2 \cos t - t.g_2^2(t) \sin t}{g_1(t).g_2(t)}, \frac{t.g_1^2(t) \cos t + b^2 \sin t}{g_1(t).g_2(t)}, -abt}{g_1(t).g_2(t)} \right),
\]

\[
\mathbf{n}_2 = \left( \frac{-b(t \cos t + \sin t)}{g_2(t)}, \frac{b(t \sin t - \cos t)}{g_2(t)}, \frac{at^2}{g_2(t)} \right),
\]
where $g_1(t) = \sqrt{b^2 + a^2 t^2}$, $g_2(t) = \sqrt{a^2 t^4 + b^2 (1 + t^2)}$. Then we compute the Euclidean curvature $\kappa$ and the Euclidean torsion $\tau$ of $\gamma$ as follows:

$$\kappa = \frac{a \sqrt{a^2 t^4 + b^2 (1 + t^2)}}{\sqrt{b^2 + a^2 t^2}}, \quad \tau = \frac{b(2 + t^2)}{a^2 t^4 + b^2 (1 + t^2)}.$$

Applying (10) we obtain the shape curvature $\tilde{\kappa}$ and the shape torsion $\tilde{\tau}$ of $\gamma$

$$\tilde{\kappa} = \frac{t(3a^2 b^2 - b^4 + a^4 t^4)}{a \sqrt{a^2 t^4 + b^2 (1 + t^2)^3}}, \quad \tilde{\tau} = \frac{b(2 + t^2) \sqrt{b^2 + a^2 t^2}}{a \sqrt{a^2 t^4 + b^2 (1 + t^2)^3}}.$$

From above equalities for $\kappa$, $\tau$ and $\tilde{\kappa}$ it follows that the focal curvatures $c_1$ and $c_2$ of $\gamma$ are

$$c_1 = \frac{1}{\kappa} = \frac{\sqrt{b^2 + a^2 t^2}}{a \sqrt{a^2 t^4 + b^2 (1 + t^2)}}, \quad c_2 = \frac{t(3a^2 b^2 - b^4 + a^4 t^4)}{ab(2 + t^2) \sqrt{a^2 t^4 + b^2 (1 + t^2)^3}}.$$

Using (12) we find the parametric representation of the focal curve $C_\gamma$ of the cylindrical curve $\gamma$

$$C_\gamma(t) = \frac{a^2 + b^2}{2 + t^2} \left( \frac{2 \cos t - t \sin t}{a}, \frac{2 \sin t + t \cos t}{a}, \frac{t^3}{b} \right), \quad t \in \mathbb{R}..$$

The curve $\gamma$ lies on the cylindrical surface

$$S(u, v) = (a \cos u + u \sin u), a(\sin u - u \cos u), bv), \quad (u, v) \in \mathbb{R}^2.$$
The normal vector \((S_u \times S_v) \gamma = (S_u \times S_v)_{u=v=t} = ab(t \sin t, -t \cos t, 0)\) to this surface along \(\gamma\) is not perpendicular to the binormal vector \(\dot{\gamma} \times \ddot{\gamma}\) of \(\gamma\) because \(\langle \dot{\gamma} \times \ddot{\gamma}, (S_u \times S_v) \gamma \rangle = -a^2b^2t \neq 0\) for \(t \neq 0\). This means that the curve \(\gamma\) is not a geodesic on the cylindrical surface \(S(u, v)\). Note that the cylindrical curve \(\gamma\) over the involute of a circle lies also on the hyperboloid of one sheet \(H\) implicitly defined by \(\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{b^2} = 1\). For \(a = 2.0\) and \(b = 5.0\) the involute of a circle (in green), the corresponding cylindrical curve \(\gamma\) (in red) on the hyperboloid \(H\), the focal curve (in blue) of \(\gamma\) are plotted from left to right in Fig. 1.

4 A general construction of a cylindrical curve over a plane curve

The results from the previous two section can be extended to a wider class of cylindrical curves.

**Theorem 4.1.** Let \(\alpha(t) = (x(t), y(t)), \quad t \in I \subset \mathbb{R}\) be regular plane curve of class \(C^3\) wit nonzero signed curvature, and let \(f(t) \in C^3\) be a real-valued function. Suppose that \(\vec{e}_3\) is the unit vector on \(Oz\)-axis and

\[
\gamma(t) = \alpha(t) + f(t) \vec{e}_3, \quad t \in I
\]

is a parameterized space curve. Then, \(\gamma(t)\) is a regular curve whose curvature and torsion are given by

\[
\kappa = \frac{\sqrt{\langle \dot{\alpha}, J\dot{\alpha} \rangle^2 + \langle \ddot{f}\dot{\alpha} - \dot{f}\ddot{\alpha}, \dddot{f}\dot{\alpha} - \ddot{f}\dddot{\alpha} \rangle}}{\sqrt{\langle \dot{\alpha}, \dot{\alpha} \rangle + \dot{\dot{f}}^2}^3} \tag{13}
\]

\[
\tau = \frac{\ddot{f}\langle \dddot{\alpha}, J\dot{\alpha} \rangle + \dddot{f}\langle -J\dot{\alpha}, \dddot{\alpha} \rangle + \dddot{f}\langle J\dddot{\alpha}, \dddot{\alpha} \rangle}{\langle \dddot{\alpha}, J\dot{\alpha} \rangle^2 + \langle \dddot{f}\dot{\alpha} - \ddot{f}\dddot{\alpha}, \dddot{f}\dot{\alpha} - \ddot{f}\dddot{\alpha} \rangle} \tag{14}
\]

**Proof:** The condition for a nonzero signed curvature of \(\alpha\) is equivalent to \(\dot{x}\dot{y} - \ddot{x}\ddot{y} \neq 0\) for any \(t \in I\). Then the binormal vector of \(\dot{\gamma}\)

\[
\dot{\gamma} \times \ddot{\gamma} = (\dddot{f}\dot{y} - \ddot{f}\dddot{y}, \dddot{f}\dddot{x} - \ddot{f}\dddot{x}, \dot{x}\dddot{y} - \dddot{x}\dot{y})
\]

is nonzero for any \(t \in I\). This implies that \(\gamma\) is a regular space curve and

\[
||\dot{\gamma} \times \ddot{\gamma}|| = \sqrt{(\dddot{f}\dot{y} - \ddot{f}\dddot{y})^2 + (\dddot{f}\dddot{x} - \ddot{f}\dddot{x})^2 + (\dot{x}\dddot{y} - \dddot{x}\dot{y})^2} = \sqrt{\dddot{f}^2\langle \dddot{\alpha}, \dddot{\alpha} \rangle - 2\dddot{f}\dddot{f}\langle \dddot{\alpha}, \dddot{\alpha} \rangle + \dddot{f}^2\dddot{f}\dddot{f}\dddot{f}\dddot{f}\dddot{f}\dddot{f}\dddot{f}\dddot{f}\dddot{f}}.
\]
Clearly, the norm of tangent vector \( \dot{\gamma} \) is \( ||\dot{\gamma}|| = \sqrt{\langle \dot{\gamma}, \dot{\gamma} \rangle + \ddot{f}^2} \). Replacing the above expressions of \( ||\dot{\gamma}|| \) and \( ||\dot{\gamma} \times \ddot{\gamma}|| \) in the first formula of (6) we get (13).

The triple vector product of \( \dot{\gamma}, \ddot{\gamma} \) and \( \dddot{\gamma} \) can be represented by

\[
\det(\dot{\gamma}, \ddot{\gamma}, \dddot{\gamma}) = \langle \dot{\gamma} \times \ddot{\gamma}, \dddot{\gamma} \rangle = (\dot{f}\dddot{y} - \dddot{f}\dot{y})\dddot{x} + (\dddot{f}\dddot{x} - \dot{f}\dddot{x})\dddot{y} + (\dddot{x}\dddot{y} - \dddot{y}\dddot{x})\dddot{f}.
\]

Then, using the formula for torsion in (6) we obtain (14).

Let \( \alpha(t) = (ae^{ct}\cos t, ae^{ct}\sin t), \ a > 0, \ t \in \mathbb{R} \) be a logarithmic spiral and let \( f(t) = be^{ct}, \ b > 0, \ c > 0, \ t \in \mathbb{R} \) be a real-valued function. Consider a cylindrical curve over logarithmic spiral

\[
\gamma(t) = (ae^{ct}\cos t, ae^{ct}\sin t, be^{ct}),
\]

where \( a, b \) and \( c \) are positive real constants.

First, we observe that \( \gamma(t) \) is a regular space curve. In fact, the vector

\[
\dot{\gamma} \times \ddot{\gamma} = a(-abce^{2ct}(c\cos t - \sin t) \), -abce^{2ct}(c\sin t + \cos t), a^2(1 + c^2)e^{2ct})
\]

is nonzero for any \( t \). The arc-length function of \( \gamma \) is given by

\[
s_\gamma(t) = \frac{\sqrt{a^2 + (a^2 + b^2)c^2e^{2ct}}}{c}.
\]

Furthermore, the unit principle normal vector \( \mathbf{n}_1 \) and the unit binormal vector \( \mathbf{n}_2 \) of \( \gamma \) are

\[
\mathbf{n}_1(t) = \left( -\frac{\cos t + c\sin t}{\sqrt{1 + c^2}}, \frac{c\cos t - \sin t}{\sqrt{1 + c^2}}, 0 \right),
\]

\[
\mathbf{n}_2(t) = \frac{-bc}{a^2(1 + c^2) + b^2c^2}\left( c\cos t - \sin t, \frac{c\cos t + c\sin t}{\sqrt{1 + c^2}}, -a\sqrt{1 + c^2} bc \right).
\]

From \( \langle \mathbf{n}_1, \mathbf{e}_3 \rangle = 0 \) it follows that the curve \( \gamma \) given by (15) is a slant helix.

Second, using (6) we calculate the Euclidean curvature and the Euclidean torsion of \( \gamma \): \( \kappa_1^\gamma = \kappa_2^\gamma = \frac{a\sqrt{1 + c^2}e^{-ct}}{a^2 + (a^2 + b^2)c^2}, \ \kappa_1^\gamma = \kappa_2^\gamma = \frac{bc e^{-ct}}{b^2c^2 + a^2(1 + c^2)}. \)

Third, according to (10) the shape curvature and the shape torsion of \( \gamma \) are

\[
\kappa_1^\gamma = \kappa_2^\gamma = \frac{c\sqrt{a^2 + (a^2 + b^2)c^2}}{a\sqrt{1 + c^2}}, \quad \kappa_2^\gamma = \kappa_2^\gamma = \frac{bc}{a\sqrt{1 + c^2}}.
\]

Hence, the curve \( \gamma \) possesses a constant shape curvature and a constant shape torsion. Such a kind of a space curve is called self-similar. Other representations of self-similar curves are given in ([3]) and ([4]). The focal curvatures \( c_1^\gamma = \frac{1}{\kappa_1^\gamma} \) and \( c_2^\gamma = \frac{\kappa_1^\gamma}{\kappa_2^\gamma} \) can be expressed as

\[
c_1^\gamma = \frac{(a^2 + (a^2 + b^2)c^2)e^{ct}}{a\sqrt{1 + c^2}}, \quad c_2^\gamma = \frac{(a^2 + (a^2 + b^2)c^2)^{3/2}e^{ct}}{ab\sqrt{1 + c^2}}.
\]
Thus, the parametrization of the focal curve $C_{\gamma}(t) = \gamma(t) + c_1 \mathbf{n}_1 + c_2 \mathbf{n}_2$ of $\gamma$ is

$$C_{\gamma}(t) = \left( a^2 + b^2 \right) \left( -\frac{c^2 e^{ct} \cos t}{a}, -\frac{c^2 e^{ct} \sin t}{a}, \frac{(1 + c^2) e^{ct}}{b} \right). \quad (17)$$

The curve $\gamma$ given by (15) lies on the cylindrical surface $S_1(u,v) = (ae^{cu} \cos u, ae^{cu} \sin u, bv)$. The normal vectors to this surface at the points of $\gamma$ are $(S_1u \times S_1v)_{\gamma} = abe^{cu}(\cos u + c \sin u, \sin u - c \cos u, 0)_{u=t,v=e^{ct}} = abe^{ct}(\cos t + c \sin t, \sin t - c \cos t, 0)$. From $\langle \dot{\gamma} \times \ddot{\gamma}, (S_u \times S_v)_{\gamma} \rangle = 0$ for any $t$, it follows that $\gamma$ is a geodesic on the surface $S_1(u,v)$. The curve $\gamma$ given by (15) lies also on the conoid $S_2(u,v) = (au \cos v, au \sin v, be^{cv})$ and on the cone $S_3: \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{b^2} = 0$. It is easy to see that $\gamma$ is not a geodesic of $S_2(u,v)$ and $S_3$. For $a = 0.5$, $b = 1.0$ and $c = 0.1$ the logarithmic spiral, the corresponding self-similar curve $\gamma$ on the conoid $S_2(u,v)$, and the focal curve $C_\gamma$ are plotted in Fig. 2.

The focal curve $C_{\gamma}$ of $\gamma$ has many properties which are analogous to the properties of the self-similar curve $\gamma$. From (17) it follows that the arc-length function of focal curve $C_{\gamma}$ is

$$s_{C}(t) = \frac{(a^2 + b^2) \sqrt{(1 + c^2)(a^2 + (a^2 + b^2)c^2) e^{ct}}}{ab}.$$
\[ \kappa^C = \frac{ab^2 e^{-ct}}{(a^2 + b^2) \sqrt{1 + c^2}} \left( a^2 + (a^2 + b^2) c^2 \right)^\frac{1}{2}, \quad \tau^C = \frac{a^2 be^{-ct}}{(a^2 + b^2) c (a^2 + (a^2 + b^2) c^2)}. \]

Using (10) we obtain the shape curvature and the shape torsion of \(C_\gamma\)

\[ \kappa^C = \frac{\sqrt{a^2 + (a^2 + b^2) c^2}}{b} = \frac{\kappa^\gamma}{\tau^\gamma}, \quad \tau^C = \frac{a \sqrt{1 + c^2}}{bc} = \frac{1}{\tau^\gamma}. \]

Let us summarize the obtained properties of the considered cylindrical curve \(\gamma\) and its focal curve \(C_\gamma\).

**Theorem 4.2.** The cylindrical curve \(\gamma: \mathbb{R} \rightarrow \mathbb{E}^3\) with a parametrization (15) is a self-similar space curve whose constant nonzero shape curvature \(\bar{\kappa}^\gamma\) and constant nonzero shape torsion \(\bar{\tau}^\gamma\) are given by (16). The focal curve \(C_\gamma: \mathbb{R} \rightarrow \mathbb{E}^3\) of \(\gamma\) is also a self-similar space curve with a constant nonzero shape curvature \(\bar{\kappa}^C = \frac{\kappa^\gamma}{\tau^\gamma}\) and a constant nonzero shape torsion \(\bar{\tau}^C = \frac{1}{\tau^\gamma}\).

**Acknowledgements.** The authors are partially supported by Research Fund of Konstantin Preslavsky University of Shumen under Grant No.RD-08-281/12.03.2015.

**References**


Received: July 9, 2015; Published: September 3, 2015