Clique Domination in a Graph*

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Abstract

Let $G$ be a nontrivial connected graph. A nonempty subset $S$ of $V(G)$ is a clique dominating set of $G$ if $S$ is a dominating set and the induced subgraph $\langle S \rangle$ of $S$ is complete. The minimum cardinality among all clique dominating sets of $G$, denoted by $\gamma_{cl}(G)$, is called the clique domination number of $G$. A clique dominating set $S$ of $G$ with $|S| = \gamma_{cl}(G)$ is called a $\gamma_{cl}$-set of $G$.

This study aims to characterize the clique dominating sets in the join, corona, composition and cartesian product of graphs and determine the corresponding clique domination number of the resulting graph.

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1 Introduction

Let \( G = (V(G), E(G)) \) be a graph with \( n = |V(G)| \) and \( m = |E(G)| \). For any vertex \( v \in V(G) \), we define the open neighborhood of \( v \) as the set \( N_G(v) = \{u \in V(G) : uv \in E(G)\} \) and the closed neighborhood of \( v \) as the set \( N_G[v] = N_G(v) \cup \{v\} \). If \( S \) is a nonempty subset of \( X \), then \( N_G(S) = \bigcup_{v \in S} N_G(v) \) and \( N_G[S] = N_G(S) \cup S \). A nonempty subset \( S \) of \( V(G) \) is a dominating set of \( G \) if for every \( v \in V(G) \setminus S \), there exists \( u \in S \) such that \( uv \in E(G) \), that is \( N_G[S] = V(G) \). The domination number of \( G \), denoted by \( \gamma(G) \), is the minimum cardinality among all dominating sets of \( G \). A dominating set \( S \) of \( G \) with \( |S| = \gamma(G) \) is called a \( \gamma \)-set of \( G \).

Let \( G \) be a nontrivial connected graph. A dominating set \( S \) of \( V(G) \) is a clique dominating set of \( G \) if the induced subgraph \( \langle S \rangle \) of \( S \) is complete. The minimum cardinality of a clique dominating set of \( G \), denoted by \( \gamma_cl(G) \), is called the clique domination number of \( G \). A clique dominating set of \( G \) with cardinality \( \gamma_cl(G) \) is called a \( \gamma_cl(G) \)-set of \( G \).

The concept of clique domination was first studied by Cozzens and Kelleher in [1]. Total domination was investigated in [2]. Domination and other variations of domination can be found in [3] and [4].

2 Results

The following are results characterizing the clique dominating sets in the join, corona, composition and cartesian product of graphs.

Remark 2.1 Let \( G \) be a connected graph. Then \( \gamma_cl(G) = 1 \) if and only if \( \gamma(G) = 1 \).

Theorem 2.2 Let \( G \) be a connected graph of order \( n \geq 4 \). Then \( \gamma_cl(G) = 2 \) if and only if \( \gamma_t(G) = 2 \) and \( \gamma(G) \neq 1 \).

Proof: Suppose \( \gamma_cl(G) = 2 \), say \( S = \{x, y\} \) is a clique dominating set of \( G \). Then \( S \) is a total dominating set of \( G \). Hence \( \gamma_t(G) = |S| = 2 \). By Remark 2.1, \( \gamma(G) \neq 1 \).

Conversely, suppose that \( \gamma_t(G) = 2 \) and \( \gamma(G) \neq 1 \). Let \( S_1 = \{a, b\} \) be a total dominating set of \( G \). Then \( S_1 \) is a clique dominating set of \( G \). Hence \( \gamma_cl(G) \leq |S_1| = 2 \). Since \( \gamma(G) \neq 1 \), \( \gamma_cl(G) \neq 1 \) by Remark 2.1. This proves that \( \gamma_cl(G) = 2 \). \( \Box \)

The join of two graphs \( G \) and \( H \), denoted by \( G + H \), is the graph with vertex-set \( V(G + H) = V(G) \cup V(H) \) and edge-set

\[
E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}.
\]
Theorem 2.3 Let $G$ and $H$ be any two graphs. A subset $S$ of $V(G + H)$ is a clique dominating set of $G + H$ if and only if one of the following statements holds:

(i) $S$ is a clique dominating set of $G$.

(ii) $S$ is a clique dominating set of $H$.

(iii) $S = S_1 \cup S_2$, where $\langle S_1 \rangle$ and $\langle S_2 \rangle$ are cliques in $G$ and $H$, respectively.

Proof: Suppose that $S$ is a clique dominating set of $G + H$. If $S \cap V(H) = \emptyset$, then $S \subseteq V(G)$ and $S$ is a clique dominating set of $G$. Similarly, if $S \subseteq V(H)$, then $S$ is a clique dominating set of $H$. Suppose $S_1 = S \cap V(G) \neq \emptyset$ and $S_2 = S \cap V(H) \neq \emptyset$. Since $\langle S \rangle$ is a clique in $G + H$, it follows that $\langle S_1 \rangle$ and $\langle S_2 \rangle$ are cliques in $G$ and $H$, respectively.

The converse is straightforward. □

Corollary 2.4 Let $G$ and $H$ be nontrivial graphs. Then

$$
\gamma_{ct}(G + H) = \begin{cases} 
1, & \text{if } \gamma(G) = 1 \text{ or } \gamma(H) = 1 \\
2, & \text{otherwise.}
\end{cases}
$$

Let $G$ and $H$ be graphs of orders $n$ and $m$, respectively. The corona $G \circ H$ of $G$ and $H$ is the graph obtained by taking one copy of $G$ and $n$ copies of $H$, and then joining the $i$th vertex of $G$ to every vertex of the $i$th copy of $H$. For every $v \in V(G)$, denote by $H^v$ the copy of $H$ whose vertices are attached one by one to the vertex $v$. Denote by $v + H^v$ the subgraph of the corona $G \circ H$ corresponding to the join $\langle \{v\} \rangle + H^v$.

The following discussion leads to the characterization of the clique dominating set in the corona of graphs.

Consider the corona of graphs $G$ and $H$ as shown below.

![Figure 1: $G \circ H$ without clique dominating set](image)
Observe that even if $G$ and $H$ have clique dominating sets, $G \circ H$ may not have any clique dominating set. On the other hand, if $G$ is complete, that is, $G = K_4$, it is easy to check that $G$ is a clique dominating set of $G \circ H$. In fact, the existence of the clique dominating set of the corona of two graphs, say $G_1$ and $G_2$ relies on the completeness of $G_1$ as exemplified in the following theorem.

**Theorem 2.5** Let $G$ be a connected nontrivial graph and $H$ be any nontrivial graph. Then $G \circ H$ has a clique dominating set $S$ if and only if $G$ is complete and $S = V(G)$.

*Proof:* Suppose that $S$ is a clique dominating set of $G \circ H$. Since $G$ is a nontrivial graph, it follows that $S \cap V(H^v) = \emptyset$ for each $v \in V(G)$. Thus, $S = V(G)$ since $S$ is a dominating set of $G \circ H$. Since $\langle S \rangle$ is a clique, it follows that $G$ is a complete graph.

The converse is easy. □

**Corollary 2.6** Let $G$ be a complete nontrivial graph and $H$ be any graph. Then

$$\gamma_d(G \circ H) = |V(G)|.$$ 

The lexicographic product $G[H]$ of two graphs $G$ and $H$ is the graph with vertex-set $V(G[H]) = V(G) \times V(H)$ and edge-set $E(G[H])$ satisfying the following conditions: $(x, u)(y, v) \in E(G[H])$ if and only if either $xy \in E(G)$ or $x = y$ and $uv \in E(H)$.

Observe that a subset $C$ of $V(G[H]) = V(G) \times V(H)$ can be written as $C = \bigcup_{x \in S} (x \times T_x)$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for every $x \in S$. Henceforth, we shall use this form to denote any subset $C$ of $V(G[H]) = V(G) \times V(H)$.

**Theorem 2.7** Let $G$ and $H$ be connected nontrivial graphs. Then $G[H]$ has a clique dominating set if and only if $G$ has a clique dominating set.

*Proof:* Suppose that $C = \bigcup_{x \in S} \{x\} \times T_x$ is a clique dominating set of $G[H]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$. Let $x, y \in S$ such that $x \neq y$. Pick any $t_1 \in T_x$ and $t_2 \in T_y$. Since $\langle C \rangle$ is complete and $(x, t_1), (y, t_2) \in C$, where $(x, t_1) \neq (y, t_2)$, it follows that $(x, t_1)(y, t_2) \in E(G[H])$. By the definition of composition of graphs, $xy \in E(G)$. Therefore, $\langle S \rangle$ is complete. Assume that $z \in V(G) \setminus S$. Choose any $t \in V(H)$. Then $(z, t) \notin C$. Since $C$ is a dominating set of $G[H]$, $(z, t)(w, q) \in E(G[H])$ for some $(w, q) \in C$. Since $z \neq w$, it follows that $zw \in E(G)$. Hence, $S$ is a dominating set of $G$. Consequently, $S$ is a clique dominating set of $G$.

Conversely, let $S$ be a clique dominating set of $G$. If $|S| = 1$, say $S = \{x\}$,
then \( S' = \{x, y\} \), where \( xy \in E(G) \) is also a clique dominating set of \( G \). Thus, we may assume that \(|S| \geq 2\). Choose any \( a \in V(H) \) and let \( T_x = \{a\} \) for each \( x \in S \). Set \( C^* = \bigcup_{x \in S} \{x\} \times T_x \neq S \times \{a\} \). Since \( \langle S \rangle \) is a clique in \( G \), \( \langle C^* \rangle \) is a clique in \( G[H] \). Let \((z, b) \notin C^*\). Suppose that \( z \notin S \). Since \( S \) is a dominating set of \( G \), there exists \( y \in S \) such that \( zy \in E(G) \). Hence, \((y, a) \in C^* \) and \((z, b)(y, a) \in E(G[H]) \). If \( z \in S \), then there exists \( w \in S \) such that \( zw \in E(G) \) and \( b \neq a \). Thus, \((z, b)(w, a) \in E(G[H]) \). This shows that \( C^* \) is a clique dominating set of \( G[H] \). \(\square\)

**Corollary 2.8** Let \( G \) and \( H \) be connected nontrivial graphs. If

\[
C = \bigcup_{x \in S} \{x\} \times T_x
\]

, where \( S \subseteq V(G) \) and \( T_x \subseteq V(H) \) for each \( x \in S \), is a clique dominating set of \( G[H] \), then \( S \) is a clique dominating set of \( G \).

**Theorem 2.9** Let \( G \) and \( H \) be connected nontrivial graphs such that \( G \) has a clique dominating set. A subset \( C = \bigcup_{x \in S} \{x\} \times T_x \), where \( S \subseteq V(G) \) and \( T_x \subseteq V(H) \) for each \( x \in S \), is a clique dominating set of \( G[H] \) if and only if \( S \) is a clique dominating set of \( G \) such that

(i) \( \langle T_x \rangle \) is a clique in \( H \) for each \( x \in S \) and

(ii) \( T_x \) is a dominating set of \( H \) whenever \( S = \{x\} \).

**Proof:** Suppose that \( C = \bigcup_{x \in S} \{x\} \times T_x \) is a clique dominating set of \( G[H] \). Then \( S \) is a clique dominating set of \( G \) by Corollary 2.8. Let \( x \in S \) and let \( a, b \in T_x \), where \( a \neq b \). Since \( \langle C \rangle \) is a clique in \( G[H] \), \((x, a)(x, b) \in E(G[H]) \). This implies that \( ab \in E(H) \). Thus, \( \langle T_x \rangle \) is a clique in \( H \). Now suppose that \( S = \{x\} \). Let \( c \in V(H) \setminus T_x \). Since \( C \) is a dominating set of \( G[H] \) and \((x, c) \notin C \), it follows that there exists \((x, d) \in C \cap N_{G[H]}((x, c)) \). Thus, \( d \in T_x \cap N_H(c) \). Therefore, \( T_x \) is a dominating set of \( H \).

For the converse, suppose that \( S \) is a clique dominating set of \( G \) satisfying (i) and (ii). Then, clearly, \( C = \bigcup_{x \in S} \{x\} \times T_x \) induces a clique in \( G[H] \). Let \((z, d) \notin C \). If \( z \notin S \), then there exists \( w \in S \) such that \( wz \in E(G) \). Choose any \( q \in T_w \). Then \((w, q) \in C \cap N_{G[H]}((z, d)) \). Suppose \( z \in S \). If \(|S| \geq 2\), then there exists \( y \in S \cap N_G(z) \). Pick any \( p \in T_y \). Then \((y, p) \in C \cap N_{G[H]}((z, d)) \). Suppose \( S = \{z\} \). Then, by assumption, \( T_z \) is a dominating set of \( H \). Hence, there exists \( t \in T_z \cap N_H(d) \). This implies that \((z, t) \in C \cap N_{G[H]}((z, d)) \). Therefore, \( C \) is a clique dominating set of \( G[H] \). \(\square\)
Corollary 2.10 Let $G$ and $H$ be connected nontrivial graphs such that $G$ has a clique dominating set. Then
\[
\gamma_{cd}(G[H]) = \begin{cases} 
1, & \text{if } \gamma(G) = \gamma(H) = 1 \\
2, & \text{if } \gamma(G) = 1 \text{ and } \gamma(H) \neq 1 \\
\gamma_{cd}(G), & \text{if } \gamma(G) \neq 1.
\end{cases}
\]

Proof: Clearly, $\gamma(G[H]) = \gamma_{cd}(G[H]) = 1$ if and only if $\gamma(G) = \gamma(H) = 1$. Suppose that $\gamma(G) = 1$ and $\gamma(H) \neq 1$. Let $S = \{x\}$ be a clique dominating set of $G$. Choose any $y \in (V(G) \setminus \{x\}) \cap N_G(x)$. Then $S_1 = \{x, y\}$ is a clique dominating set of $G$. Let $a \in V(H)$ and set $T_x = T_y = \{a\}$. Then by Theorem 2.9, $C = S_1 \times \{a\}$ is a clique dominating set of $G[H]$. Thus, $\gamma_{cd}(G[H]) = |C| = |S_1| = 2$. Next, suppose that $\gamma(G) \neq 1$. Let $S$ be a $\gamma_{cd}$-set of $G$. Then $|S| \geq 2$. Choose any $a \in V(H)$ and let $T_x = \{a\}$ for each $x \in S$. Then $C = \bigcup \limits_{x \in S} ([x] \times T_x) = S \times \{a\}$ is a clique dominating set of $G[H]$ by Theorem 2.9. Hence, $\gamma_{cd}(G[H]) \leq |C| = |S| = \gamma_{cd}(G)$. On the other hand, if $C^* = \bigcup \limits_{x \in S^*} ([x] \times T_x)$ is a $\gamma_{cd}$-set of $G[H]$, then $S^*$ is a clique dominating set of $G$. Thus, $\gamma_{cd}(G[H]) = |C^*| \geq |S^*| \geq \gamma_{cd}(G)$. Therefore, $\gamma_{cd}(G[H]) = \gamma_{cd}(G)$. □

The Cartesian product $G \square H$ of two graphs $G$ and $H$ is the graph with $V(G \square H) = V(G) \times V(H)$ and $(u, u')(v, v') \in E(G \square H)$ if and only if either $uv \in E(G)$ and $u' = v'$ or $u = v$ and $u'v' \in E(H)$.

Note that if $C \subseteq V(G \times H)$, then the $G$-projection and $H$-projection of $C$ are, respectively, the sets
\[
C_G = \{u \in V(G) : (u, b) \in C \text{ for some } b \in V(H)\}
\]
and
\[
C_H = \{v \in V(H) : (a, v) \in C \text{ for some } a \in V(G)\}.
\]

Remark 2.11 Let $G$ and $H$ be connected nontrivial graphs. If $C$ is a dominating set of $G \square H$, then $C_G = V(G)$ or $C_H = V(H)$.

It follows from Remark 2.11 that if $C$ is a dominating set of $G \square H$, then either $C = \bigcup \limits_{x \in V(G)} ([x] \times T_x)$ or $C = \bigcup \limits_{a \in V(H)} [D_a \times \{a\}]$, where $T_x \subseteq V(H)$ for each $x \in V(G)$ and $D_a \subseteq V(G)$ for each $a \in V(H)$.

Theorem 2.12 Let $G$ and $H$ be connected nontrivial graphs of orders $m$ and $n$, respectively. Then $G \square H$ has a clique dominating set if and only if either $G$ is complete and $\gamma(H) = 1$ or $H$ is complete and $\gamma(G) = 1$. Moreover,
\[
\gamma_{cd}(G \square H) = \begin{cases} 
m, & \text{if } G \text{ is complete and } \gamma(H) = 1 \\
n, & \text{if } H \text{ is complete and } \gamma(G) = 1 \\
\min\{m, n\}, & \text{if } G \text{ and } H \text{ are both complete.}
\end{cases}
\]
Proof: Suppose that $G \boxtimes H$ has a clique dominating set, say $C$. Also, let $C_G = V(G)$. Then $C = \bigcup_{x \in V(G)} \{x\} \times T_x$, where $T_x \subseteq V(H)$ for each $x \in V(G)$.

Let $y, z \in V(G)$ such that $y \neq z$. Pick any $a \in T_y$ and $b \in T_z$. Since $\langle C \rangle$ is complete and $(y, a)$ and $(z, b)$ are distinct elements of $C$, it follows that $(y, a)(z, b) \in E(G \boxtimes H)$. Hence, $yz \in E(G)$ and $a = b$. This implies that $G$ is complete and that $T_x = \{a\}$ for all $x \in V(G)$ and for some $a \in V(H)$. Now, let $c \in V(H) \setminus T_x$. Since $C$ is a dominating set of $G \boxtimes H$ and $a \neq c$, it follows that $ac \in E(H)$. This shows that $T_x = \{a\}$ is a dominating set of $H$. Thus, $\gamma(H) = 1$. Similarly, $H$ is complete and $\gamma(G) = 1$ if $C_H = V(H)$. The converse is easy.

Therefore,

$$
\gamma_{cl}(G \boxtimes H) = \begin{cases} 
  m, & \text{if } G \text{ is complete and } \gamma(H) = 1 \\
  n, & \text{if } H \text{ is complete and } \gamma(G) = 1 \\
  \min\{m, n\}, & \text{if } G \text{ and } H \text{ are both complete.}
\end{cases}
$$

This proves the assertion. $\square$

References


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