On Vertex Denominators of the Boolean Quadric Polytope Relaxation

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Abstract

We study the $M_n$ and $M_{n,3}$ relaxations of the boolean quadric polytope $BQP_n$. Corresponding via the covariance mapping cut polytope relaxations are known as the rooted semimetric and metric polytopes. A sequence of $M_{n,3}$ fractional vertices, generated by triangle inequalities, is constructed with the largest denominator growing exponentially in $n$. This refutes the previously conjectured linear upper bound for this value.

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Keywords: polyhedral combinatorics, boolean quadric polytope, metric polytope, fractional vertices.

1 Boolean quadric polytope and its relaxations

We consider the well-known boolean quadric polytope $BQP_n$ [8], satisfying the constraints

\[ x_i + x_j - x_{i,j} \leq 1, \]
\[ x_{i,j} \leq x_i, \]  
\[ x_{i,j} \leq x_j, \]
\[ x_{i,j} \geq 0, \]
\[ x_i, x_{i,j} \in \{0, 1\}, \]
for all \( i, j : 1 \leq i < j \leq n \).

Polytope \( BQP_n \) is constructed from the NP-hard problem of unconstrained boolean quadratic programming:

\[
Q(x) = x^T Q x \to \text{max},
\]

where vector \( x \in \{0, 1\}^n \), and \( Q \) is an upper triangular matrix, by introducing new variables \( x_{i,j} = x_i x_j \).

It also should be noted that \( BQP_n \) is in one-to-one correspondence via the covariance mapping with the well-known cut polytope \( CUT_{n+1} \) \[6\].

If we exclude from the system (1) – (5) the constraints (5) that the variables are integral, the remaining system (1) – (4) describe the boolean quadric relaxation polytope \( M_n \). Corresponding cut polytope relaxation is known as the rooted semimetric polytope, and we keep the metric names for considered polytopes, as they have the same properties.

If we introduce the additional variables

\[
x_{i,j}^{1,1} = x_{i,j}, \quad x_{i,j}^{2,2} = 1 - x_{i,i},
\]

\[
x_{i,j}^{1,2} = x_{j,j} - x_{i,j}, \quad x_{i,j}^{2,1} = x_{i,i} - x_{i,j},
\]

\[
x_{i,j}^{2,2} = 1 - x_{i,i} - x_{j,j} + x_{i,j},
\]

the rooted semimetric polytope \( M_n \) can be written in a homogeneous form as follows

\[
x_{i,j}^{1,1} + x_{i,j}^{1,2} + x_{i,j}^{2,1} + x_{i,j}^{2,2} = 1, \quad (6)
\]

\[
x_{i,j}^{1,1} + x_{i,j}^{1,2} = x_{k,j}^{1,1} + x_{k,j}^{1,2}, \quad (7)
\]

\[
x_{i,j}^{1,1} + x_{i,j}^{2,1} = x_{i,l}^{1,1} + x_{i,l}^{2,1}, \quad (8)
\]

\[
x_{i,i}^{1,1} = x_{i,i}^{2,1} = 0, \quad (9)
\]

\[
x_{i,j}^{1,1} \geq 0, \quad x_{i,j}^{1,2} \geq 0, \quad x_{i,j}^{2,1} \geq 0, \quad x_{i,j}^{2,2} \geq 0, \quad (10)
\]

where \( 1 \leq k \leq i \leq j \leq l \leq n \) \[3\].

Such a complex description of \( M_n \) allows us to consider more sophisticated objective functions. For example in [1] it is shown that the special instance of the MAX-CUT problem of the following form: for a given directed weighted graph \( G = (V, E) \) it is required to find a partition of the vertex set \( V \) into two disjoint subsets \( P \) and \( Q \) to maximize the value

\[
\sum_{i \in P, j \in Q} C_{i,j} - \sum_{i \in Q, j \in P} C_{i,j},
\]

is polynomially solvable by linear programming on \( M_n \).

Points of the rooted semimetric polytope \( M_n \) in the form (6) – (10) can be conveniently represented as a block upper triangular matrix (Table 1).
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Since the rooted semimetric polytope $M_n$ is a relaxation of $BQP_n$, it has additional fractional vertices. The good news is that they are completely described and their coordinates only take values from the set $\{0, \frac{1}{2}, 1\}$ [8].

We consider, following [2] (a similar construction is considered in [4]), a sequence $M_{n,k}$ of nested relaxations of the boolean quadric polytope, obtained by augmenting the system (6)–(10) by the $BQP_k$ constraints.

The first relaxation, different from the rooted semimetric polytope, is $M_{n,3}$. It is obtained by imposing the triangle inequalities that describe the $BQP_3$ facets [8]:

\[
\begin{align*}
  x_{i,i} &+ x_{j,j} + x_{k,k} - x_{i,j} - x_{i,k} - x_{j,k} \leq 1, \\
-x_{i,i} &+ x_{i,j} + x_{i,k} - x_{j,k} \leq 0, \\
-x_{j,j} &+ x_{i,j} - x_{i,k} + x_{j,k} \leq 0, \\
-x_{k,k} &- x_{i,j} + x_{i,k} + x_{j,k} \leq 0.
\end{align*}
\]

for all $i, j, k$, where $1 \leq i < j < k \leq n$.

Corresponding relaxation of the cut polytope is known as the metric polytope. In the new form, we can merge all new inequalities into one class:

**Theorem 1.1.** Metric polytope $M_{n,3}$ is defined by the system of inequalities (6) – (10) and the additional constraints

\[
\sum_{i \in \{k, l, m\}} x_{i,i}^{a_i a_i} - \sum_{i,j \in \{k, l, m\}, i < j} x_{i,j}^{a_i a_j} \leq 1,
\]

for all $k, l, m$, where $1 \leq k < l < m \leq n$ and all vectors $(a_k, a_l, a_m) \in [1, 2]^3$.

**Proof.** Inequality (11) with respect to new coordinates already has the required form (15) and corresponds to the vector $a = (1, 1, 1)$. Constraints (12) – (14) can be easily transformed to (15), e.g.

\[
\begin{align*}
-x_{i,i}^{1,1} &+ x_{i,j}^{1,1} + x_{i,k}^{1,1} - x_{j,k}^{1,1} \leq 0, \\
x_{i,i}^{1,1} + x_{i,j}^{2,2} - 1 \leq 0, \\
x_{i,j}^{2,2} &- 1 + x_{i,j}^{1,1} + x_{i,k}^{1,1} - x_{j,k}^{1,1} \leq 0, \\
x_{i,j}^{1,1} + x_{i,j}^{1,2} = x_{j,j}^{1,1}, \\
x_{i,k}^{1,1} + x_{i,k}^{1,2} = x_{k,k}^{1,1}, \\
x_{i,j}^{2,2} + x_{i,j}^{1,1} + x_{k,k}^{1,1} - x_{i,k}^{1,2} - x_{i,k}^{1,1} - x_{j,k}^{1,1} \leq 1.
\end{align*}
\]
Thus, the inequality (12) corresponds to the vector \( a = (2, 1, 1) \). Besides, each of the constraints (11) – (14) generates a symmetrical constraint

\[
x_{i,j}^1 + x_{i,j}^1 + x_{j,k}^2 - x_{i,j}^1 - x_{i,k}^1 - x_{j,k}^1 \leq 1,
\]

\[
1 - x_{i,j}^2 + 1 - x_{j,k}^2 + 1 - x_{i,k} - x_{i,j}^1 - x_{i,k}^1 - x_{j,k}^1 \leq 1,
\]

\[
(1 - x_{i,j}^1) + (1 - x_{i,k}^1) + (1 - x_{j,k}^1) - x_{i,k}^2 - x_{j,k}^2 - x_{k,k}^2 \leq 1,
\]

\[
x_{i,j}^1 + x_{i,j}^2 + x_{i,k}^2 + x_{i,k}^2 + x_{j,k}^2 + x_{j,k}^2 + x_{j,k}^2 + x_{j,k}^2 - x_{i,j}^1 - x_{i,k}^1 - x_{j,k}^1 \leq 1,
\]

\[
x_{i,j}^2 = x_{i,j}^1 + x_{i,j}^1, \quad x_{j,k}^2 = x_{j,k}^1 + x_{j,k}^1, \quad x_{k,k}^2 = x_{i,k}^1 + x_{i,k}^1,
\]

\[
x_{i,j}^1 + x_{i,k}^1 + x_{j,k}^1 \leq 1,
\]

\[
(x_{j,k}^2 - x_{i,j}^2) + (x_{i,j}^2 - x_{i,k}^2) + (x_{i,k}^2 - x_{j,k}^2) \leq 1,
\]

\[
x_{i,j}^2 + x_{j,k}^2 + x_{j,k}^2 - x_{i,j}^2 - x_{i,k}^2 - x_{j,k}^2 \leq 1.
\]

\[\square\]

Inequalities (15) are somewhat redundant, since each constraint is included twice, but due to the high symmetry are simple to use.

## 2 Metric polytope vertex denominators

Triangle inequalities are powerful enough to cut off all nonintegral vertices of the rooted semimetric polytope \( M \). Thus, polytopes \( M_n \) and \( M_{n,3} \) do not have common fractional vertices. However, the triangle inequalities generate new fractional vertices of \( M_{n,3} \) that have a more complex structure than vertices of the rooted semimetric polytope \( M_n \). So in [8] there was considered a special class of graphic vertices of the \( M_{n,3} \). It was proved that every graphic vertex of \( M_{n,3} \) has its denominator less or equal to \( 2(n - 2) \) and listed a series of nonintegral vertices with denominators growing linearly in \( n \). It was conjectured [9] that the denominators of the \( M_{n,3} \) fractional vertices are bounded above by a linear function.

Since all vertices of \( M_{n,3} \) are known for \( n \leq 8 \) [5], we can see that the denominators may have more diverse values. Indeed, the largest known denominator of \( M_{n,3} \) vertices is 39.

We will show that the lower bound on the largest denominator of the metric polytope vertices is at least exponential in \( n \).

**Theorem 2.1.** Denote by \( d(M_{n,3}) \) the largest denominator of the coordinates of \( M_{n,3} \) vertices, then \( \forall n \geq 4 : \)

\[
d(M_{n,3}) \geq 3 \cdot 2^\left\lfloor \frac{n-4}{4} \right\rfloor.
\]
Proof. We construct a series of the metric polytope vertices that satisfy the theorem.

We assume \( n = 4 + 3s + q \), where \( s, q \in \mathbb{Z}^+ \). Consider a point \( u \in M_n \) satisfying the following conditions for all \( p \in \{0, 1, 2, \ldots, s - 1\} \):

- for all \( 1 \leq i < j \leq 4 + 3s \):
  \[
  x_{i,j}^{2,1} = 0 \text{ if } i = 4 + 3p, \quad (16) \\
  x_{i,j}^{2,2} = 0 \text{ if } i \neq 4 + 3p; \quad (17)
  \]

- for all \( 1 \leq k < l < m \leq 4 \):
  \[
  \sum_{i \in \{k,l,m\}} x_{i,i}^{2,2} - \sum_{i,j \in \{k,l,m\}, i < j} x_{i,j}^{2,2} = 1; \quad (18)
  \]

- for all \( k = 4 + 3p \) and all \( 5 + 3p \leq l < m \leq 7 + 3p \):
  \[
  \sum_{i \in \{k,l,m\}} x_{i,i}^{a_k,a_l,a_m} - \sum_{i,j \in \{k,l,m\}, i < j} x_{i,j}^{a_k,a_l,a_m} = 1, \quad (19)
  \]
  where \((a_k, a_l, a_m) = (1, 2, 2)\);

- for \( 4 + 3s + 1 \leq i \leq 4 + 3s + q \):
  \[
  x_{i,i}^{2,2} = 0. \quad (20)
  \]

First, we show that the system of linear equations (6) – (9), (16) – (20) has a unique solution. If \( n \) is equal to 4, we can easily solve the system and obtain the solution presented in Table 2. It satisfies the constraints of \( M_{4,3} \), so it’s a fractional vertex. In fact, it is well known \( M_{4,3} \) vertex [5, 7].

Now let \( k = 4 + 3p \) (for some \( p \in \{0, 1, 2, \ldots, s - 1\} \)) and consider the blocks \( k \leq i \leq j \leq k + 3 \) (Table 3).

From the equations (16), (17) and (19) we obtain the system

\[
\begin{align*}
  x_{k,k}^{1,1} + x_{k+1,k+1}^{2,2} + x_{k+2,k+2}^{2,2} &= 1, \\
  x_{k,k}^{1,1} + x_{k+1,k+1}^{2,2} + x_{k+3,k+3}^{2,2} &= 1, \\
  x_{k,k}^{1,1} + x_{k+2,k+2}^{2,2} + x_{k+3,k+3}^{2,2} &= 1.
\end{align*}
\]

As \( x_{k,k}^{1,1} = 1 - x_{k,k}^{2,2} \), we have

\[
x_{k+1,k+1}^{2,2} = x_{k+2,k+2}^{2,2} = x_{k+3,k+3}^{2,2} = \frac{1}{2} x_{k,k}^{2,2}.
\]
Thus, for every three block columns denominator of the coordinates is multiplied by two. Since we start with the denominator equal to 3 for the first four columns, we obtain the required lower bound

\[ d(M_{n,3}) \geq 3 \cdot 2 \left\lfloor \frac{n-4}{3} \right\rfloor. \]

All the remaining coordinates are the solutions of the system (6) – (9), (20). Vertex of the polytope \( M_{8,3} \), constructed according the procedure, is presented in Table 4.

Considering point clearly belongs to the rooted semimetric polytope \( M_n \). It remains to verify that it satisfies the constraints (15) and therefore is a vertex of the metric polytope.

First of all, the inequality

\[ \sum_{i \in \{k,l,m\}} x_{i,i}^{2,2} - \sum_{i,j \in \{k,l,m\}, i < j} x_{i,j}^{2,2} \leq 1 \]
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Table 4: Fractional vertex of the $M_{8,3}$ polytope.

<table>
<thead>
<tr>
<th>$x_{i,i}$</th>
<th>$x_{i,j}$</th>
<th>$x_{i,k}$</th>
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<tbody>
<tr>
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is easily true for all $1 \leq i \leq n$ since $x_{i,i}^2 \leq \frac{1}{3}$. Taking into account the double-counting of inequalities, the same is true for

$$\sum_{i \in \{k,l,m\}} x_{i,i}^1 = \sum_{i,j \in \{k,l,m\}, i < j} x_{i,j}^1 \leq 1.$$ 

So we have to check the rest triangle inequalities for all $1 \leq i < j < k \leq n$. Consider the inequality

$$x_{i,i}^2 + x_{j,j}^2 + x_{k,k}^1 - x_{i,j}^2 - x_{i,k}^2 - x_{j,k}^2 \leq 1.$$ 

Note that if $x_{i,j}^2 \neq 0$, then $x_{i,j}^1 = 0$ (16), and $x_{i,j}^2 = x_{j,j}^2$:

$$(x_{i,i}^2 + x_{k,k}^1 - x_{i,k}^2) - x_{j,k}^2 = x_{i,k}^1 + x_{i,k}^2 + x_{i,k}^2 - x_{j,k}^2 \leq 1 - x_{j,k}^2 \leq 1.$$ 

And if $x_{i,j}^2 = 0$, then $x_{i,k}^2 = 0$ (17), and $x_{i,k}^1 = x_{i,k}^1$, again

$$x_{j,j}^2 + x_{k,k}^1 - x_{j,k}^2 = x_{j,k}^1 + x_{j,k}^2 + x_{j,k}^2 \leq 1.$$
Now let’s turn to the inequality
\[ x_{i,j}^{2,2} + x_{k,k}^{2,2} - x_{i,j}^{1,2} + x_{i,k}^{2,2} - x_{j,k}^{2,1} \leq 1. \]
As before \( x_{i,k}^{2,2} = 0 \), otherwise it is satisfied immediately, and we have
\[ x_{i,k}^{2,2} = 0 \Rightarrow x_{i,j}^{2,2} = 0 \Rightarrow x_{i,j}^{1,2} = x_{i,j}^{2,2}, \]
\[ x_{j,j}^{1,2} + x_{k,k}^{2,2} - x_{j,k}^{2,1} = x_{j,j}^{1,1} + x_{j,k}^{2,1} + x_{j,k}^{2,2} \leq 1. \]

Regarding the last inequality
\[ x_{i,i}^{1,1} + x_{j,j}^{2,2} + x_{k,k}^{2,2} - x_{i,i}^{2,1} - x_{i,k}^{2,2} - x_{j,k}^{2,2} \leq 1, \]
we can without loss of generality assume
\[ x_{i,j}^{2,1} = x_{i,k}^{2,1} = x_{j,k}^{2,2} = 0, \]
otherwise the inequality holds as before. Since \( x_{i,j}^{2,1} = 0 \), we have \( i = 4 + 3p \) (16), and for all \( k > j > i \)
\[ x_{k,k}^{2,2} \leq x_{j,j}^{2,2} \leq \frac{1}{2} x_{i,i}^{2,2}, \]
\[ x_{i,i}^{1,1} + x_{j,j}^{2,2} + x_{k,k}^{2,2} \leq x_{i,i}^{1,1} + x_{i,i}^{2,2} = 1. \]

Point, constructed by the described procedure, satisfies all the triangle inequalities, and, as a unique solution of the system of linear equations (6)–(9), (16)–(20), is a fractional vertex of the \( M_{n,3} \) polytope.

\[ \square \]

Note that via the covariance mapping [6]
\[ y_{i,n+1} = x_{i,i} \] for all \( 1 \leq i \leq n \),
\[ y_{i,j} = x_{i,i} + x_{j,j} - 2x_{i,j} \] for all \( 1 \leq i < j \leq n \),
coordinates \( y_{i,j} \) of the metric polytope \( MET_{n+1} \) are obtained from the coordinates \( x_{i,j} \) of the boolean quadric polytope relaxation \( M_{n,3} \), thus the same property of the denominators exponential growth holds for the metric polytope \( MET_{n+1} \). For example, the vertex from Table 4 transforms into the vertex of \( MET_9 \):
\[ \frac{1}{6}(4, 4, 4, 3, 3, 3, 2, 4, 4, 4, 4, 3, 3, 3, 2, 4, 4, 3, 3, 2, 4, 1, 1, 1, 2, 4, 2, 2, 1, 5, 2, 1, 5, 1, 5, 6). \]
And if we make another step of the algorithm and assume \( s = 2 \), we construct the fractional vertex of the metric polytope \( MET_{11} \) with denominator 12:
\[ \frac{1}{12}(8, 8, 8, 8, 6, 6, 6, 5, 5, 5, 5, 8, 8, 8, 8, 6, 6, 6, 5, 5, 5, 8, 8, 8, 6, 6, 6, 5, 5, 5, 8, 2, 2, 2, 3, 3, 8, 4, 4, 3, 3, 3, 10, 4, 3, 3, 3, 10, 1, 1, 1, 1, 10, 2, 2, 11, 2, 11, 2, 11, 11). \]
Presented lower bound is far from being accurate; however, it refutes the conjecture of the denominators linear growth [9], and shows that they may have more diverse values.

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References


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