Incomplete Markets Equilibrium Relying on 2-Dimensional Lattice-Subspaces

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Abstract

This paper is devoted to the detailed proof that incomplete markets equilibrium is generically existent for a set of economies of full measure in the two-date, finite-state model. The proof is independent from the dimension of the (incomplete) market, while it mainly relies on the utility functions of the individuals (investors) which quantify the preference on both of shares of the riskless asset and a state-discriminating replicated payoff.

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1. INTRODUCTION

This paper is devoted to the detailed proof that incomplete markets equilibrium is generically existent for a set of economies of full measure in the two-date, finite-state model. The proof is independent from the dimension of the (incomplete) market, while it mainly relies on the utility functions of the individuals (investors) which quantify the preference on both of shares of the riskless asset and a state-discriminating replicated payoff. The seminal reference for Equilibrium in Incomplete Markets is [8]. The model in which Equilibrium in Incomplete Markets is developed in [8], is also extensively presented in [20, Ch.11]. The frame of study relies on regular values of maps between smooth manifolds. The difference between this approach and the one
of the present paper is that the present paper proves the existence of portfolio equilibrium, relying on the Maximum Theorem of Berge, see [1, Th.17.31]. Consequently, the portfolio-price duality relying on portfolio dominance partial ordering for finite-dimensional portfolio spaces, (see [2, p.273-74]) is used. Also, in [2, p.272], the definition of the no-arbitrage portfolio price being used in this paper is given. It is easy to verify that a portfolio equilibrium implies the existence of a General Equilibrium with Incomplete Markets. Finally, the form of the market clearing conditions we propose by the definition of portfolio allocation (see Def. 2.1) is equivalent to the validity of the form of market clearing conditions mentioned in [20, p.310] - in the case of the strongly resolving markets we consider. Other references on the same theme are the papers [15], [10]. These papers assume that the individuals do have a specific attitude towards the uncertainty indicated by the set of S states of the world, which involves probability vectors in the definition of their preference relations. Our paper does not make such an assumption, since the equivalent definition is rather a geometric one. More specifically, the utility functions are defined by the preferences of the individuals (investors) only on portfolios of a two-dimensional lattice-subspace, independently from the other elements of the asset span X. This paper is organized as follows: First, we present some essential notions from ordered spaces' theory, expressed in the case of finite dimensions. In the next, we revisit the finite-state finance and we insist on the role of positive bases and lattice-subspaces in it. The rest part of the paper is devoted to the proof the main result, which is the existence of portfolio equilibrium in incomplete markets which are strongly resolving and they include the riskless asset. A great part is devoted to the repetition of the proof of a recent equilibrium result of ours for excess demand correspondences, which implies the normalization of the portfolio prices, through bases of the portfolio dominance cone (see [2, p.273-74]. Also, since their completion by options F(X) is actually the complete market \( \mathbb{R}^S \), the state-discriminating replicated payoffs are a set of full measure in the subspace X of \( \mathbb{R}^S \). This fact implies that the utility functions of the investors may be quantified on any of these payoffs and on the riskless asset, as mentioned above.

1.1. Finite-dimensional ordered linear spaces. Let \( E \) be a Euclidean space. A set \( C \subseteq E \) satisfying \( C + C \subseteq C \) and \( \lambda C \subseteq C \) for any \( \lambda \in \mathbb{R}_+ \) is called wedge. A wedge for which \( C \cap (-C) = \{0\} \) is called cone. If \( \geq \) is a binary relation on \( E \) satisfying the following properties:

(i) \( x \geq x \) for any \( x \in E \) (reflexive)

(ii) If \( x \geq y \) and \( y \geq z \) then \( x \geq z \), where \( x, y, z \in E \) (transitive)

(iii) If \( x \geq y \) then \( \lambda x \geq \lambda y \) for any \( \lambda \in \mathbb{R}_+ \) and \( x + z \geq y + z \) for any \( z \in E \), where \( x, y \in E \) (compatible with the linear structure of \( E \)),

then the pair \( (E, \geq) \) is called partially ordered linear space. If \( E = \mathbb{R}^m_+ = \{x \in \mathbb{R}^m | x(i) \geq 0, i = 1, 2, \ldots, m\} \) then this set is a cone of \( E \) and the binary relation \( x \geq y \iff x(i) \geq y(i), i = 1, 2, \ldots, m \) is called component-wise partial ordering.
or usual partial ordering of $\mathbb{R}^m$. $(\mathbb{R}^m_+, \geq)$ is a partially ordered linear space. The set $P = \{x \in E | x \geq 0\}$ is called (positive) wedge of the partial ordering $\geq$ of $E$. Given a wedge $C$ in $E$, the binary relation $\geq_C$ defined as follows:

$$x \geq_C y \iff x - y \in C,$$

is a partial ordering on $E$, called partial ordering induced by $C$ on $E$. If the partial ordering $\geq$ of the space $E$ is antisymmetric, namely if $x \geq y$ and $y \geq x$ implies $x = y$, where $x, y \in E$, then $P$ is a cone. The usual partial ordering of $\mathbb{R}^m_+$ is antisymmetric, its positive cone is $\mathbb{R}^m_+$ and the partial ordering induced by $\mathbb{R}^m_+$ on $\mathbb{R}^m$ is actually the usual partial ordering. In $\mathbb{R}^m$ the topological and the algebraic dual coincide. The partially ordered vector space $(\mathbb{R}^m_+, \geq)$ is a vector lattice under the usual partial ordering $\geq$, namely for any $x, y \in \mathbb{R}^m_+$, both the supremum and the infimum of $\{x, y\}$ with respect to this partial ordering, exist in $\mathbb{R}^m$. If $F$ is a subspace of a vector lattice $\mathbb{R}^m_+$ and the partial ordering induced on $F$ by the cone $F_+ = F \cap E_+$ makes $F$ a vector lattice, then $F$ is called lattice-subspace. Then for any $x, y \in F$, $\sup_F \{x, y\} = x \vee_F y$, $\inf_F \{x, y\} = x \wedge_F y$ exist in $F$. Their relation to equivalent $x \vee y, x \wedge y \in E$ is the following:

$$x \wedge_F y \leq x \wedge y \leq x \vee y \leq x \vee_F y.$$ 

If $D$ is a subspace of $\mathbb{R}^m$ having a basis $(b_i)_{i=1,2,...,k}$, this basis is called positive basis if and only if

$$D_+ = D \cap \mathbb{R}^m = \{x = \sum_{i=1}^k \lambda_i b_i | \lambda_i \geq 0, i = 1, 2, ..., k\}.$$ 

Choquet-Kendall Theorem [16, Pr.1.1] refers to the connection between finite-dimensional vector lattices and positive bases: A finite-dimensional ordered vector space $E$ with a closed and generating cone $E_+$ is a vector lattice if and only if has a positive basis. Also in I.A. Polyakov [16, Th.3.6], the determination of such a positive basis in the case of a finite-dimensional lattice-subspace of $C(\Omega)$, where $\Omega$ is some compact and Hausdorff topological space, is provided. Hence, Theorem [16, Th.3.6] is also applicable on $\mathbb{R}^m$. A vector $f \in \mathbb{R}^m$ is a positive functional of a cone $C$ in $\mathbb{R}^m$ if and only if $f(x) \geq 0, x \in C$, while it is a strictly positive functional of $C$ if and only if $f(x) > 0, x \in C \setminus \{0\}$. If $D$ is a subspace of $\mathbb{R}^m$ and $g \in \mathbb{R}^m$ is a strictly positive functional of $D \cap C$, or else a strictly positive functional of $X$ with respect to the partial ordering induced by $C$ on $D$, we say that $f \in \mathbb{R}^m$ is a strictly positive extension of $g$ if and only if $f$ is a strictly positive functional of $C$ and $g(x) = f(x), x \in D$. A positive projection of $\mathbb{R}^m$ being partially ordered by a cone $C$ on the subspace $D$ being partially ordered by the cone $D \cap C$ is a projection $P : \mathbb{R}^m \rightarrow D$, for which $P(x) \in D \cap C$ for any $x \in C$. A strictly positive projection is a positive projection which has the additional property: $P(x) = 0, x \in C \iff x = 0$. As I.A. Polyakov proved in [16, Th.3.4], the finite-dimensional lattice-subspaces
in $C(\Omega)$ having positive bases with nodes, where $\Omega$ is some compact and Hausdorff topological space, are examples of ranges of positive projections. Hence, Theorem [16, Th.3.4] is also applicable on $\mathbb{R}^m$.

1.2. Finite-State Finance. Suppose that there are two periods of economic activity and $S$ states of the world. At time-period $t = 0$ there is uncertainty about the true state of the world, while at time-period $t = 1$ this state is revealed. Suppose that there are $n$ primitive assets in the market which are non-redundant, namely their payoff vectors $x_1, x_2, ..., x_n \in \mathbb{R}^S$ at time period $t = 1$, are linearly independent. A portfolio in this market is a vector $\theta = (\theta_1, \theta_2, ..., \theta_n)$ of $\mathbb{R}^n$ in which $\theta_i, i = 1, 2, ..., n$ denotes the units invested to the asset $i$. If $\theta_i \geq 0$, then the investment to $\theta_i$ units of the asset $i$ denotes a long position on these units. If $\theta_i < 0$, then the investment to $-\theta_i$ units of the asset $i$ denotes a short position on $-\theta_i$ units of the asset $i$. The payoff of a portfolio $\theta$, if the payoff vectors $x_1, x_2, ..., x_n \in \mathbb{R}^S$ are expressed in terms of the numeraire as well, is the vector $T(\theta) = \sum_{i=1}^n \theta_i x_i$. The range of the operator $T : \mathbb{R}^n \to \mathbb{R}^S$ is called asset span of the market, derived by $x_1, x_2, ..., x_n$ and in the rest of the paper will be denoted either by $[x_1, x_2, ..., x_n]$, or by $X$. By $X$ we will also denote the $S \times n$-matrix, whose columns are the vectors $x_1, x_2, ..., x_n$. In this paper we suppose that $n < S$, hence the market of $x_1, x_2, ..., x_n$ is incomplete. A contingent claim is any liability $c \in \mathbb{R}^S$, while a derivative is a contingent claim whose payoff is connected through a functional form to some portfolio payoff for the asset span of $x_1, x_2, ..., x_n$. If for some contingent claim $c$ there is some portfolio $\theta$ such that $T(\theta) = c$, then the contingent claim $c$ is called replicated or hedged (by the portfolio $\theta$). Any portfolio $\theta \in \mathbb{R}^n$ such that $T(\theta) = c$ is called replicating portfolio or hedging portfolio of $c$. Classical examples of derivatives are (European) options, which include the corresponding call options and put options. Call and put options written on some asset $c$ under some risky strike vector $u$, different from 1. If we denote such a vector by $u$, In this case, the call option written on $c$ with strike price $a$ with respect to $u$ is the contingent claim whose payoff vector is $(c - au)^+$. In the same way, the corresponding put option is $(au - c)^+$. The last call option is denoted by $c_u(c, a)$, while the put option is denoted by $p_u(c, a)$. The call option $c_u(c, a)$ and put option $p_u(c, a)$ are called non-trivial if $c_u(c, a) > 0, p_u(c, a) > 0$, respectively. This definition implies that for both of these vectors all of their components are positive and at least one of them is non-zero.

It is well-known that the completion by options $F_1(X)$ of the asset span $X = [x_1, x_2, ..., x_n]$ with respect to 1 is the vector subspace of $\mathbb{R}^S$ which contains all the derivatives written on the elements of the asset span $X$, see [12]. It is also well-known (see [12, Th.3]) that the completion $F_1(X)$ of $X$ by options is the sublattice $S(Y)$ generated by $Y = [X \cup \{1\}]$.

Since $F_1(X)$ is a sublattice and hence a lattice-subspace, it has a positive basis

$$\{b_i\}_{i=1,2,..,\mu}, \dim F_1(X) = \mu.$$
This positive basis is a \textit{partition of the unit} (see \cite{12}). Its elements are \textit{binary vectors}, (see also \cite{4, 12}). The determination of this positive basis relies on \cite[Th.3.7]{17}

According to \cite[Def.18]{12}, a vector \( e \in F_u(X) \) is an \( F_u(X) \)-efficient fund if \( F_u(X) \) is the linear subspace of \( \mathbb{R}^S \) which is generated by the set of nontrivial call options and the set of non-trivial put options of \( e \).

We also remind of the statements of \cite[Th.19]{12}, \cite[Pr.20]{12}, \cite[Th.21]{12}, respectively:

\begin{itemize}
\item[(i)] Suppose that \( \{b_1, b_2, \ldots, b_u\} \) is a positive basis of \( F_u(X) \), \( u = \sum_{i=1}^{u} \lambda_i b_i \), and \( \lambda_i > 0 \) for each \( i \). Then the vector \( e = \sum_{i=1}^{u} \kappa_i b_i \) of \( F_u(X) \) is an \( F_u(X) \)-efficient fund if and only if \( \frac{\alpha_i}{\lambda_i} \neq \frac{\beta_j}{\lambda_j} \) for each \( i \neq j \).
\item[(ii)] Each non-efficient subspace of \( F_u(X) \) is a proper sublattice of \( F_u(X) \).
\item[(iii)] Suppose that \( \{b_1, b_2, \ldots, b_u\} \) is a positive basis of \( F_u(X) \) and that \( u = \sum_{i=1}^{u} \lambda_i b_i \) with \( \lambda_i > 0 \) for each \( i \). Then:
\end{itemize}

\begin{itemize}
\item[(i)] the nonempty set \( D = Y \setminus \bigcup_{i \in I}(Y \cap Z_i) \), where \( \{Z_i \mid i \in I\} \) is the set of non-efficient subspaces of \( F_u(X) \), is the set of \( F_u(X) \)-efficient funds of \( Y \) and the Lebesgue measure of \( Y \) is supported on \( D \)
\item[(ii)] \( F_u(X) \) is the subspace of \( \mathbb{R}^S \) generated by the set of the call options \( \{c_u(x,a) \mid x \in Y, a \in \mathbb{R}\} \) written on the elements of \( Y \). If \( u \in X, F_u(X) \) is the subspace \( X_1 \) of \( \mathbb{R}^S \) generated by the set of call options \( O_1 = \{c_u(x,a) \mid x \in X, a \in \mathbb{R}\} \) written on the elements of \( X \).
\end{itemize}

\textbf{Lemma 1.1.} There exists an efficient fund \( e \in X_+, e > 0 \) with respect to the strike vector \( 1 \).

\begin{proof}
Direct from \cite[Th.21]{12}.
\end{proof}

We recall the exact definition of resolving and strongly resolving markets.

\textbf{Definition 1.2.} (see also \cite[p.20]{12}) An asset span \( X = [x_1, x_2, \ldots, x_n] \) is \textbf{resolving} if and only if \( 1 \in X \) and for any pair of different states \( s_1 \neq s_2 \), a primitive asset \( x_k, k = 1, 2, \ldots, n \) exists, such that \( x_k(s_1) \neq x_k(s_2) \).

\textbf{Definition 1.3.} (see also \cite[p.21]{12}) An asset span \( X = [x_1, x_2, \ldots, x_n] \) is \textbf{strongly resolving} if and only if \( 1 \in X \) and for any choice of \( n \) different states and a contingent claim \( y \in F_i(X) \), a unique portfolio \( \theta \in \mathbb{R}^n \) exists, such that \( X \cdot \theta = x \) and \( y \) coincide on these \( n \) states of the world.

It is well-known, see \cite{3}, that strongly resolving markets are actually a subset of the resolving markets.

We also have the following

\textbf{Proposition 1.4.} If we suppose that the vectors of the date-1 payoffs of the primitive assets \( x_1, x_2, \ldots, x_n \) are linearly independent and \( 1 \in X \), then \( F_1(X) = \mathbb{R}^S \), where \( X = [x_1, x_2, \ldots, x_n] \), except a set of vectors \( x_1, x_2, \ldots, x_n \) of Lebesgue measure zero in \( \mathbb{R}^S \).
Proof. In the last part of [12], we gave a brief proof about the fact that resolving markets have the property $F_1(X) = \mathbb{R}^S$. It is also well-known that resolving matrices are in general position, namely the complement of the set of them is a null-set in the vector space of the matrices $S \times n$, whose entries are real numbers. Hence the super-set of all the $S \times n$-matrices (markets), such that $1 \in X = [x_1, x_2, ..., x_n]$ where $x_1, x_2, ..., x_n$ are linearly independent and they have the property that $F_1(X) = \mathbb{R}^S$ are also in general position. □

If $F$ is a subspace of a vector lattice $E$ and the partial ordering induced on $F$ by the cone $F_+ = F \cap E_+$ makes $F$ a vector lattice, then $F$ is called lattice-subspace. Then for any $x, y \in F$, $\sup_F \{x, y\} = x \vee_F y, \inf_F \{x, y\} = x \wedge_F y$ exist in $F$. Their relation to equivalent $x \wedge y, x \vee y \in E$ is the following:

$$x \wedge_F y \leq x \wedge y \leq x \vee y \leq x \vee_F y,$$

in terms of the partial ordering of $E$. About lattice -subspaces and their influence in economics see [16, 17, 2, 12].

**Theorem 1.5.** The subspace $L = [e, 1]$ created by a 'generically existent’ $e \in X_+$ and the riskless asset $1$ is a lattice -subspace of $\mathbb{R}^S$.

**Proof.** According to the Choquet-Kendall Theorem, see [16, Pr.1.1] Theorem, we have to prove that $L_+ = L \cap \mathbb{R}_+^S$ is closed and generating. $L_+$ is closed, since the component functionals of the basis $\{e, 1\}$ are continuous. $L_+$ is generating, since $\mathbb{R}_+^S$ is generating, because it contains the order unit $1$, which is also an order unit for $L$ under the induced ordering implied by $L_+$. Since $L$ is a closed subspace of $\mathbb{R}^S$ it is also a complete space under the induced topology, hence by Baire Category the order unit $1$ is also an interior point of $L_+$, which implies that $L_+$ is generating. □

**Corollary 1.6.** $L$ has a positive basis.

**Proof.** Direct from [16, Pr.1.1], [16, Th.3.6] and 1.5. □

2. **Incomplete Markets Equilibrium Revisited**

We consider an exchange economy in which a finite number of individuals exist, $j = 1, 2, ..., I$. Each of them has an initial wealth $\omega_j = (\omega_{j0}, \omega_{j1})$, where $\omega_{j0} > 0$ denotes $j$-individual’s wealth at time-period 0, while $\omega_{j1} = (\omega_{j1}(1), \omega_{j1}(2), ..., \omega_{j1}(S)) \in \mathbb{R}_+^S$ denotes the contingent wealth of $j$ at time-period 1. This definition implies that $\omega_j \in \mathbb{R}_+^{S+1}$. The total wealth is denoted by $\omega = \sum_{j=1}^I \omega_j \in \mathbb{R}_+^{S+1}$, while by $\omega$ we denote the vector $(\omega_1, \omega_2, ..., \omega_I)$. A consumption vector is some $y = (y_0, y_1) \in \mathbb{R}_+^{S+1}$, while by $y_0 \in \mathbb{R}_+, y_1 \in \mathbb{R}_+$. The portfolio price vector at time period 0 is equal to $q = (q_1, q_2, ..., q_n)$, where $q_i > 0$ denotes the amount of Euros or in general of the numeraire payable, needed for any individual to take the long position at one unit of the security.
Increasing, continuous and strictly concave transformation of
following:
\[ B_j (\omega_j; q) = \{ y_j \in \mathbb{R}^{S+1}_+ | y_{j0} - \omega_{j0} = -q \cdot \theta^j; y_{j1} - \omega_{j1}) = X \cdot \theta^j, \theta \in \mathbb{R}^n \}. \]
Market clearing equations take the form \[ \sum_{j=1}^I y_j = \sum_{i=1}^I \omega_j \iff \sum_{j=1}^I \theta^j = 0. \]

**Definition 2.1.** A portfolio allocation is a vector in the set
\[ A_X = \{ \theta = (\theta^1, \theta^2, ..., \theta^J) \in (\mathbb{R}^n)^J | \sum_{j=1}^J X \cdot \theta^j = 0, \omega^j + X \cdot \theta^j \in \mathbb{R}^{S+1}_+ \} \]

**Proposition 2.2.** Every \( \theta \in A_X \) satisfies the market clearing equations if \( q \) is a no-arbitrage portfolio price and the asset span is strongly resolving (including the riskless asset \( 1 \in \mathbb{R}^S \)).

**Proof.** It suffices to prove that if \( X \) is a strongly resolving \( S \times n \) matrix, such that \( 1 < n < S \), then
\[ X \cdot \theta = 0 \iff \theta = 0. \]
The solution set \( S \) of the linear system \( X \cdot \theta = 0 \) is equal to \( \cap_{k=1}^d C_k \), \( d = B(S, n) \) - being the \( n + 1 \)-th coefficient of the expansion \( (1 + t)^S = \sum_{m=0}^S B(S, m)t^m \), while \( C_k \) denotes the solution set:

1. Determine the \( n \) states out of the \( S \) states of the world, which correspond to the choice of the \( k \)-th combination
2. Solve the corresponding \( n \times n \) linear system
3. Determine which of the solutions of the previous \( n \times n \) system satisfy the rest \( S - n \) linear equations.
4. These solutions are the set \( C_k \)

For the specific case, since every \( n \times n \) submatrix of \( X \) is non-singular (\( X \) is strongly resolving), the only solution is 0, which belong to any \( C_k, k = 1, 2, ..., d \) from the definition of strongly resolving matrices. Hence, \( S = \{0\} \). \( \square \)

Individuals’ utilities may be quantified either on consumption vectors, or on portfolios. We are going to derive existence of equilibrium for the above model, by using the results of the previous Section.

**Theorem 2.3.** Utility functions of the individuals may be extracted from the linear utility both on the riskless asset and for an efficient fund \( e \in X_+ \), if market \( X \) is strongly resolving, given the vector \( w \).

**Proof.** Since by Theorem 1.5 \( L = [e, 1] \) is a lattice -subspace, we may define the strictly positive projection operator \( P_{\omega_j}(\theta^j) = \frac{1}{2} [x^*_1(\theta^j)b_1 + x^*_2(\theta^j)b_2] \) where \( \{b_1, b_2\} \) is the normalized positive basis of \( L \) and \( x^*_i(x), i = 1, 2 \) are the component functionals of it, given that the wealth of the \( j \)-individual is \( \omega_j \) and the portfolio that she selects at time period 0 is \( \theta^j \). By \( P_{\omega_j} \) we define \( A_{\omega_j}(\theta^j) = \frac{1}{2} [x^*_1(\theta^j) + x^*_2(\theta^j)] \). Then we may define the \( u_j \) as an increasing, continuous and strictly concave transformation of \( A_{\omega_j} \) as follows:
$u_j(\omega_j(\theta^j)) = g_j(A_{\omega_j}(\theta^j))$, where $g_j$ is an increasing, continuous, concave function $g_j : \mathbb{R}_+ \to \mathbb{R}$. Hence we get the indirect utility

$$v_{j1,\omega_j}(y_1) = u_j(\omega_{j1}) + u_{j,\omega_j}(\theta^j), j = 1, 2, ..., I.$$  

\[ \Box \]

**Remark 2.4.** The utility functions of the individuals $j = 1, 2, ..., J$ as they are defined in 2.1 are actually **portfolio utility functions**, for any $w$.

**Theorem 2.5.** Under utilities of Theorem 2.3, demand portfolio correspondence for any $j = 1, 2, ..., I$ and aggregate portfolio demand correspondence are well-defined and upper -hemicontinuous (continuous for almost all markets in the sense of Lebesgue measure).

**Proof.** The budget set correspondence

$$(\omega^j, q, X) \mapsto B(\omega^j, q, X) = \{ y^j \in \mathbb{R}^{S+1}_+ \mid y^j_0 - \omega^j_0 = -q \cdot \theta^j y^j_1 - \omega^j_1 = X \cdot \theta^j, \theta^j \in \mathbb{R}^n \}$$

is convex valued, since every budget set is the intersection of two convex sets: the positive cone $\mathbb{R}^{S+1}_+$ and the transition of the subspace $\langle W(q, X) \rangle$ (see [14, Th.9.2]), at the point $\omega^j$. Every such set is also compact, since $q$ is a no-arbitrage portfolio price (see [14, Th.9.2]). Since each of these sets is compact in terms of $y^j$, namely closed and bounded, we will prove that it is compact in terms of portfolios. Since $|y^j_0| \leq M, M > 0$, this implies $q \cdot \theta^j - \omega^j_0 \leq |\omega^j - q \cdot \theta^j| \leq M$. But since $q$ is a no-arbitrage price, it is a strictly positive functional of the **portfolio dominance cone** $C = \{ \theta \in \mathbb{R}^n \mid X \cdot \theta \geq 0 \}$ (see [2]). Hence a positive real number $a > 0$ exists such that $q \theta \geq a \| \theta \|_1$ for any $\theta \in C$. This implies that every budget set is bounded in terms of portfolios. The proof of closedness of these sets is included in the proof of the fact that the budget correspondence is upper hemicontinuous in terms of portfolios. If $q_n \to q, \theta^j_n \to \theta^j$, while $\theta^j_n$ is such that $y^j_n \in B(\omega^j, q_n, X)$, then $y^j = \omega^j + X \cdot \theta^j \in B(\omega^j, q, X)$. This is true since by the continuity of the inner product $\omega^j_0 - q_n \cdot \theta^j_n \to \omega^j_0 - q \cdot \theta^j$ and the sequence $a_n = \omega^j_0 - q_n \cdot \theta^j_n$ of non-negative terms, converges to a non-negative limit $a = \omega^j_0 - q \cdot \theta^j$. The same happens for any state $s = 1, 2, ..., S$ and for the convergence of $\omega^j(s) + X \cdot \theta^j(s)$, being a sequence of non-negative real numbers to the limit $\omega^j(s) + X \cdot \theta^j(s)$. For lower hemicontinuity, consider a portfolio price sequence $q_n \to q$, for a specific $q$. Since this sequence is bounded, it has a convergent subsequence $q_{k_n} \to q$. For the sequence $k_n, n \in \mathbb{N}$, for each $\theta^j \in B(\omega^j, q, X)$, from the Axiom of Choice we consider $\theta^j_{k_n} \in B(\omega^j, q_{k_n}, X)$, such that $\theta^j_{k_n} \to \theta^j$. The last convergence arises from the budget equations

$\theta^j_{k_n} \in B(\omega^j, q_{k_n}, X), \theta^j \in B(\omega^j, q, X)$ and the fact that $q_n \to q$. Since this may be repeated for any $q$ and for any $\theta^j \in B(\omega^j, q, X)$ the budget correspondence is lower hemicontinuous in terms of portfolios. Since by Theorem 2.3, the portfolio utility functions are continuous, the portfolio demand correspondence for any individual $j = 1, 2, ..., J \theta^j(q)$ is upper hemicontinuous and compact-valued by Berge’s Maximum Theorem, see [1, Th.]. The same happens with
the aggregate portfolio demand correspondence \( z(q) = \sum_{j=1}^{J} \theta^j(q) \). Also, due to the concavity of \( g_j \), \( \theta^j(q) \) is convex-valued.

\[ \square \]

**Definition 2.6.** A portfolio equilibrium is consisted by a portfolio allocation (a vector of portfolios \( (\theta^1, \theta^2, \ldots, \theta^J) \) such that \( \sum_{j=1}^{J} \theta^j = 0 \)), and a price \( q \) such that \( \theta^j = \theta^j(q) \), \( j = 1, 2, \ldots, J \).

In the next we will use the following well-known

**Theorem 2.7.** (Bipolar Theorem) If \( K \) is a wedge of \( \mathbb{R}^m \), then \( K^00 = (K^0)^0 = K \).

**Proposition 2.8.** The set of strictly positive functionals \( K' \) of \( K \), where \( K \) is a closed cone of \( \mathbb{R}^m \) is non-empty.

**Theorem 2.9.** ([18, Pr.3]) Every strictly positive functional of a closed cone \( K \) of \( \mathbb{R}^m \), defines a bounded base on \( K \).

The above are also essential for the next.

**Theorem 2.10.** Consider a closed cone in \( \mathbb{R}^m \) and a base of it \( B_f \) for some strictly positive functional of it. Any upper hemicontinuous correspondence \( z : B_f \rightarrow 2^{\mathbb{R}^m} \) with non-empty, convex, compact values satisfies the weak Walras law:

\[
\text{If } q \in B_f \text{ then there exists } z \in z(q) \text{ such that } q \cdot z \leq 0.
\]

Then the set of equilibrium prices \( \{ q \in B_f \mid z(q) \cap C^0 \} \) is compact and non-empty.

**Proof.** The set of strictly positive functionals of \( C \) is non-empty in this case (from Proposition 2.8). Hence the existence of a strictly positive functional \( f \) of \( C \), that defies a closed, bounded base \( B_f \) of \( C \) is assured. Then, we may follow the lines of the proof of the equilibrium theorem in [9], as it is also presented in [5, Th.18.17], according to the Browder’s Selection Theorem (see in [7, Th.1]). First of all the set of equilibrium prices is a closed subset of the base \( B_f \), from the upper hemicontinuity of \( z \). But since \( B_f \) is a compact set of \( \mathbb{R}^m \), the set of equilibrium prices is also a compact set. In order to prove that it is a non-empty set, we suppose that the set is empty, hence for any \( p \in B_f \), there exists a functional \( k \neq 0 \) such that

\[ k \cdot z > 0 \geq k \cdot g, \quad z \in z(p), \quad g \in C^0, \]

, since \( C^0 \) is a wedge. Also, since \( k \neq 0 \), it may taken to lie on the base \( B_f \). Hence, a new correspondence is defined by \( k : B_f \rightarrow 2^{\mathbb{R}^p} \), such that \( p \mapsto K(p) = \{ h \in B_f \mid h \cdot z > 0 > h \cdot g, z \in z(p), g \in C^0 \} \). We remind the Selection Theorem of Browder: Let \( E \subset \mathbb{R}^n \) be a convex set and \( r : E \rightarrow 2^{\mathbb{R}^d} \) has convex values and \( r^{-1}(y) \) is open for any \( y \). Then there is a continuous selection \( f : E \rightarrow \mathbb{R}^d, \text{ such that } f(x) \in r(x), x \in E \). \( k \) has non-empty, convex values for any \( p \). Also, if \( p \in K^{-1}(h) \), this implies \( h \cdot z > 0 > h \cdot g, z \in z(p), g \in C^0 \), where also
we may define another correspondence $\phi : B_f \to 2^{\mathbb{R}^m}$, with $\phi(p) = C^0, p \in B_f$

Since $z$ is upper hemicontinuous and the correspondence $\phi(p) = C^0, p \in B_f$

is also upper hemicontinuous, the set $k^u(z : h \cdot z > 0) \cap \phi^u(g : h \cdot g < 0)$ is a neighborhood of $p$, contained in $K^{-1}(p)$.

Hence, by the Browder’s Selection Theorem, a continuous selection $k(p) \in K(p)$ exists, such that

$$k(p) \cdot z > 0 > k(p) \cdot g, z \in z(p), g \in C^0, p \in B_f.$$  

From Brouwer Fixed Point Theorem, there exists some fixed point of the continuous function $k : B_f \to B_f$, since $B_f$ is closed and bounded, namely compact. For this fixed point $p_0$ such that $p_0 = k(p_0), p_0 \cdot z > 0$ for any $z \in z(p_0)$, a contradiction, because weak Walras law is satisfied by $z$ for any $p \in B_f$.

Hence the set of equilibrium prices is non-empty.

\[\square\]

**Lemma 2.11.** The aggregate portfolio demand correspondence $z(q) = \sum_{j=1}^{J} \theta^j(q)$ 

is satisfies the Strong Walras law: $q \cdot z = 0, z \in z(q)$.

**Proof.** Since $q$ is a no-arbitrage portfolio price, then $q = \pi_1 \cdot X, \pi_1 \in \mathbb{R}^S_+$. Hence,

$$q \cdot z = \pi_1 \cdot X \cdot z = \pi_1 \cdot X \cdot (\sum_{j=1}^{I} \theta^j) = \pi_1 \cdot (\sum_{j=1}^{I} X \cdot \theta^j),$$

where

$$z \in z(q), z = \sum_{j=1}^{J} \theta^j, \theta^j \in \theta^j(q), j = 1, 2, ..., J$$

But from the Lemma 2.2, the sum

$$\sum_{j=1}^{I} X \cdot \theta^j = 0,$$

since $(\theta^1, \theta^2, ..., \theta^J) \in A_X$. \[\square\]

**Theorem 2.12.** If market $X$ is strongly resolving, $1 \in X$ and portfolio utilities have the form of Theorem 2.3, then a portfolio equilibrium exists.

**Proof.** The aggregate portfolio demand correspondence $z(q) = \sum_{j=1}^{J} \theta^j(q)$ is upper hemicontinuous and satisfies the Strong Walras law: $q \cdot z = 0, z \in z(q)$. We also have to prove that the portfolio dominance cone $C = \{\theta \in \mathbb{R}^n | X \cdot \theta \in \mathbb{R}^S_+\}$, is actually a cone. Since $q$ is a strictly positive functional of $C$, it defines a closed and bounded base. Since $q$ is a strictly positive functional of $C$ as a no arbitrage price, by the Bipolar Theorem, it is an interior point of $C^0$. This implies that $C^0$ is generating, hence $C^{00} = C = C$ is a cone. A strictly positive functional of $C^0$ exists and it is actually the portfolio $\theta_0 = (1, 1, ..., 1) = 1_n \in \mathbb{R}^n$. This holds because for any $d \in C^0 \setminus \{0\}, d \cdot 1_n > 0$.

This implies that we may use $\frac{1}{n} 1_n$ to be the $f$ that normalizes the portfolio
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prices in Theorem 2.10. Then the set of portfolio prices \( q_0 \in B_f \) such that 
\( z(q_0) \cap C \neq \emptyset \) is non-empty and compact. For any of the prices in the set 
\( \{ q \in B_f | z(q) \cap C \neq \emptyset \} \) we have that there exists a \( z \in z(q_0) \) such that 
\( z \in C \). If \( z = 0 \), then the portfolio allocation \( (\theta^1, \theta^2, ..., \theta^J) \) corresponding to 
\( z = \sum_{j=1}^{J} \theta^j \), where \( \theta^j \in \theta^j(q_0), j = 1, 2, ..., J \), together with \( q_0 \) is a portfolio 
equilibrium. If \( z \in C \setminus \{ 0 \} \), then we would have \( q_0 \cdot z = 0 \) from the Strong Walras Law, 
while on the other hand since \( q_0 \) is a no-arbitrage price \( q_0 \cdot z > 0 \), a contradiction. Namely, \( z = 0 \) is the only valid case. □

References


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