Some New Inequalities for a Sum of Exponential Functions

Steven G. From

Department of Mathematics
University of Nebraska at Omaha
Omaha, Nebraska 68182 USA

Abstract

In this paper, we present some new inequalities for sums of exponential functions which improve upon upper and lower estimates given in Ahmad [2004]. In some cases, a more general choice of exponential function parameters is allowed. Numerical comparisons are also made. Besides upper and lower bounds for minimum values, we present a very accurate approximation to the minimum values considered in Ahmad [2004] which is applicable to functions belonging to the absolutely monotonic class.

Keywords: Absolutely monotonic; delay differential equation; exponential functions; inequalities

1 Introduction

In Ahmad [2004], the result

\[ \min_{x > 0} \frac{e^{Tx}}{x} = Te \]  

(1)

was generalized in the form of Theorems A and B below, for \( T > 0 \):
Theorem A (Theorem 2.1 of Ahmad [2004].) Let $p > 0$, $q > 0$, and $T > 0$. Then

$$pT \exp \left( 1 + \frac{q}{pe} e^{-\sqrt{\frac{q}{pe}}} \right) \leq \min_{x > 0} \frac{pe^T x + q}{x} \leq pT \exp \left( 1 + \frac{q}{pe} \right). \quad (2)$$

Theorem B (Theorem 2.2 of Ahmad [2004].) Let $p_i > 0$, $T_i > 0$, $i = 1, \ldots, n$; $T = \max_{1 \leq i \leq n} T_i$. Then

$$L_A \equiv e \sum_{i=1}^n p_i T_i \leq \min_{x > 0} \frac{\sum_{i=1}^n p_i e^{T_i x}}{x} \leq \exp \left( 1 + \frac{\sum_{i=1}^n p_i (T - T_i)}{e \sum_{i=1}^n p_i T_i} \right) \equiv U_A. \quad (3)$$

Theorems A and B were motivated by considering characteristic equations of various delay differential equations, which often involve sums of exponential functions divided by $x$ when considering oscillation conditions for these delay differential equations. For examples of such equations and applications, see Ahmad [2004], p. 2.

2 Main Results

In this paper, we propose new upper and lower estimates/bounds and compare them both analytically and numerically to those given by (2) and (3) in Theorems A and B above. We also propose very accurate approximations to the minimum values in Theorems A and B below when bounds are not required, just very accurate approximations. We shall allow $T_i = 0$ in Theorem B. First we give some new upper bounds for

$$M_1 = \min_{x > 0} \frac{pe^T x + q}{x} \quad (4)$$

and compare them to the upper bound in Theorem A above.

Theorem 2.1 below gives a new upper bound for $M_1$ and compares it to the upper bound in Theorem A. Part c) of Theorem 2.1 says $U_1$ below is an improvement of the upper bound in Theorem A above.

Theorem 2.1

a) $M_1 \leq (pe + q)T \equiv U_1$
b) \( M_1 \leq \frac{pT}{p+q} \left( pe^{\frac{p+q}{p}} + q \right) \equiv U_1^* \)

c) \( U_1 \leq pT \exp \left( 1 + \frac{q}{pe} \right) \).

**Proof.** Let \( f(x) = \frac{pe^Tx + q}{x}, x > 0 \). Differentiation of \( f \) gives

\[
\begin{align*}
f'(x) &= x^{-2}[pe^Tx(xT - 1) - q] < x^{-2}(xT - 1)pe^Tx. \\
\end{align*}
\]

Also,

\[
\begin{align*}
f''(x) &= x^{-3} \left[ pe^Tx \cdot ((Tx - 1)^2 + 1) + 2q \right] > 0 \quad \text{for} \quad x > 0.
\end{align*}
\]

So \( f \) is convex for \( x > 0 \) and \( f' \) is strictly increasing for \( x > 0 \). Since \( f'(x^+) \to -\infty \) as \( x \to 0^+ \) and \( f \) is convex, the equation \( f'(x) = 0 \) has a unique root, call it \( x_0 \), satisfying \( x_0T - 1 \geq 0 \), by (5) above. Also,

\[
f \left( \frac{1}{T} \right) \geq f(x_0) = M_1.
\]

But \( f \left( \frac{1}{T} \right) = U_1 \). Thus, \( M_1 \leq U_1 \) and part a) is proven.

To prove b), we modify slightly the proof of part (a). Then

\[
f'(x) > x^{-2} \left[ e^Tx(xpT - (p + q)) \right] = 0
\]
when \( x = \frac{p+q}{pT} \). Using the same argument as above,

\[
U_2 = f \left( \frac{p + q}{pT} \right) \geq f(x_0) = M - 1.
\]

This proves b).

To prove c), let \( t = \frac{q}{pe} \). Then for \( t \geq 0 \), \( e^{1+t} \geq e(1+t) = e + \frac{q}{p} \), from which we obtain

\[
pT \exp \left( 1 + \frac{q}{pe} \right) \geq pT \left( e + \frac{q}{p} \right) = (pe + q)T = U_1.
\]

This completes the proof of (c) and the proof of the theorem is complete.

We shall give a new lower bound/estimate for \( M_1 \) as a special case of a more general result to be given next. This new lower bound will improve on both lower bounds given in Theorems A and B above.

First, to present more bounds, we need to discuss some results on absolutely monotonic functions.
Definition 2.2 A function $g(x)$ is absolutely monotonic on $(0, \infty)$ if $g(x)$ has the form
\[ g(x) = \int_0^\infty e^{tx} dH(t), \] (6)
where $H(t)$ is bounded and nondecreasing on $(0, \infty)$.

Remark 2.3 In Sinik [2014], it was noted that the main theorem for absolutely monotonic functions on $(0, \infty)$ from the book by Mitronovic, et al. [1993] was incorrect. Originally, an absolutely monotonic function $g(x)$ on $(0, \infty)$ was defined to have derivatives of all orders on $(0, \infty)$ with
\[ q^{(k)}(x) \geq 0, \quad 0 < x < \infty. \] (7)
But Definitions (6) and (7) are not equivalent, it turns out. However, we are interested in Definition (6) above, since the numerators of the function $f(x)$ to be minimized have form (6) above. For this definition, it is well-known that for $x > 0$,
\[ g^{(k)}(x)g^{(k+2)}(x) \geq (g^{(k+1)}(x))^2, \quad k = 0, 1, 2, \ldots. \] (8)
See, for example, Mitrinovic, Pecaric and Fink [1993], p. 366. See also Widder [1946], p. 167.

Let’s now consider upper and lower bounds for the minimum value
\[ M_2 = \min_{x > 0} f(x), \] (9)
where
\[ f(x) = \frac{g(x)}{x} \quad \text{and} \quad g(x) = \sum_{i=1}^n p_i e^{T_i x}, \] (10)
Clearly, $g(x)$ absolutely monotonic on $(0, \infty)$ and has form (6). We now present several new upper and lower bounds for $M_2$ and compare them to the bounds of Ahmad [2004] given in Theorem B, both analytically and numerically. We also allow some, but not all, of the $T_i$ values to be zero. Theorem 2.4 below presents new upper and lower bounds for $M_2$. The lower bound of Theorem B is embedded in a family of lower bounds for $M_2$, all at least as large as the lower bound of Theorem B.

Theorem 2.4 Suppose $p_i > 0$, $i = 1, 2, \ldots, n$ and $T_i \geq 0$, not all zero, $i = 1, 2, \ldots, n$. Let
\[ T_{\text{ave}} = \frac{\sum_{i=1}^n p_i T_i}{\sum_{i=1}^n p_i}, \]
and
\[ T_{\text{max}} = \max\{T_i : 1 \leq i \leq n\}. \]
Let $x^* = \frac{1}{T_{\text{max}}}$, $x^{**} = \frac{1}{T_{\text{ave}}}$. Let $0 \leq c \leq x^*$. Then we have the following:
Some new inequalities

(a) Let $h_1(c) = g(c) = \sum_{i=1}^{n} p_i e^{T_i c}$, $h_2(c) = g'(c) = \sum_{i=1}^{n} p_i T_i e^{T_i c}$. Let $A(c) = \frac{h_2(c)}{h_1(c)}$. Then $M_2 \geq h_2(c) \cdot e^{1-c \cdot A(c)} \equiv L(c)$. In addition, $L(c)$ is nondecreasing in $c$, $0 \leq c \leq x^*$. Thus, the best possible lower bound is $L(x^*)$.

(b) Let $0 \leq c \leq x^*$. Let

$$D = \frac{g'(x^{**})}{g(x^{**})} = \frac{\sum_{i=1}^{n} p_i T_i e^{T_i x^{**}}}{\sum_{i=1}^{n} p_i e^{T_i x^{**}}}.$$ 

Then $M_2 \leq D \cdot h_1(c) e^{1-cD} \equiv U(c)$.

(c) $L(c) \geq L(0) = (\sum_{i=1}^{n} p_i T_i) e = L_A$, the lower bound of Theorem B. Thus, $L(c)$ is at least as good as the lower bound of Ahmad [2004].

(d) $U(c)$ is nonincreasing in $c$, $0 \leq c \leq x^*$. Thus, the best such upper bound for $M_2$ is $U(x^*)$.

**Proof.** Let $w(x) = Ln(g(x)) = Ln \left( \sum_{i=1}^{n} p_i e^{T_i x} \right)$. Then $w(x)$ has derivatives

$$w'(x) = \frac{g'(x)}{g(x)} \quad \text{and} \quad w''(x) = \frac{g(x)g''(x) - (g'(x))^2}{(g(x))^2}, \quad x > 0.$$ 

Since $g$ is absolutely monotonic on $(0, \infty)$, $w''(x) \geq 0$, $x > 0$. Differentiating $f(x) = \frac{g(x)}{x}$, we obtain

$$f'(x) = x^{-2} \left( x \sum_{i=1}^{n} p_i T_i e^{T_i x} - \sum_{i=1}^{n} p_i e^{T_i x} \right) \quad (11)$$

$$= x^{-2} \left( \sum_{i=1}^{n} p_i (T_i x - 1) e^{T_i x} \right). \quad (12)$$

Now if $T_i = 0$, $T_i x - 1 = -1 < 0$. If $x < x^*$, $T_i x - 1 < 0$ also. Thus, $0 \leq x < x^*$ gives $f'(x) < 0$. From (11), we obtain

$$f'(x) = x^{-2} \left( \sum_{i=1}^{n} p_i \right) \cdot \sum_{i=1}^{n} Q_i (T_i x - 1) e^{T_i x}, \quad (13)$$

where $Q_i = \frac{p_i}{\sum_{k=1}^{n} p_k}$, $i = 1, 2, \ldots, n$. Note that $Q_i \geq 0$, $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} Q_i = 1$. Let $J_x(T) = (T x - 1) e^{T x}$, $x > 0$, $T > 0$. Then differentiating with respect to $T$, we obtain

$$J'_x(T) = T x^2 e^{T x} \quad \text{and} \quad J''_x(T) = (T x^3 + x^2) e^{T x} \geq 0.$$
Thus, $J_x(T)$ is a convex function of $T$ for each $x > 0$. Applying Jensen’s inequality, we obtain

$$f'(x) \geq x^{-2} \left( \sum_{i=1}^{n} p_i \right) \left( x \cdot \left( \sum_{i=1}^{n} Q_i T_i \right) - 1 \right) e^{\left( \sum_{i=1}^{n} Q_i T_i \right) x}$$

$$= 0, \text{ when } x = \frac{1}{T_{\text{ave}}} = x^{**}.$$

Thus, $x > x^{**}$ implies $f'(x) > 0$. Thus, a minimum $f(x)$ must occur on the interval $[x^*, x^{**}]$. The Mean Value Theorem or a first degree Taylor expansion gives

$$w(x) = w(c) + w'(\theta) \cdot (x - c), \quad x^* \leq x \leq x^{**} \quad (14)$$

where $\theta$ is a real number in $(c, x)$. Since $w'$ is increasing, we obtain

$$w(x) \geq w(c) + w'(c) \cdot (x - c). \quad (15)$$

Exponentiation of both sides of (15) gives, using $w'(c) = A(c)$,

$$g(x) \geq \left[ h_1(c) \cdot e^{-cA(c)} \right] e^{A(c) \cdot x}. \quad (16)$$

Thus,

$$\min_{x > 0} f(x) \geq \min_{x > 0} \frac{\left[ h_1(c) e^{-cA(c)} \right] \cdot e^{A(c) \cdot x}}{x}$$

$$= h_1(c) e^{-cA(c)} \cdot \min_{x > 0} \frac{e^{A(c) \cdot x}}{x}$$

$$= h_1(c) e^{-cA(c)} \cdot A(c) \cdot e$$

$$= h_2(c) e^{1-cA(c)} = L(c).$$

This proves part (a).

To prove (b), we use similar arguments. Then (14) gives instead

$$w(x) \leq w(c) + w'(x^{**})(x - c) \quad x^* \leq x \leq x^{**}$$

$$= w(c) + D(x - c). \quad (17)$$

Exponentiating gives

$$\min_{x > 0} f(x) \leq h_1(c) \cdot e^{-cD} \cdot \min_{x > 0} \frac{e^{Dx}}{x}$$

$$= Dh_1(c) e^{1-cD} = u(c).$$
To prove (c), consider (15). Then

\[ w(x) \equiv h_x(c) \equiv w(c) + w'(c) \cdot (x - c). \]

Fixing \( x \) in \([x^*, x^{**}]\) and differentiating the right-hand side with respect to \( c \), we obtain

\[
\begin{align*}
\frac{d}{dc} h_x(c) &= w'(c) + w'(c) \cdot (-1) + (x - c)w''(c) \\
&= (x - c)w''(c) \geq 0,
\end{align*}
\]

since \( 0 \leq c \leq x^* \) and \( x^* \leq x \leq x^{**} \) gives \( x - c \geq 0 \). Thus, the lower bound function \( h_x(c) \) considered as a function of \( c \) is nondecreasing in \( c \). Thus, \( L(c) \) is nondecreasing in \( c \) for all \( x \) in \([x^*, x^{**}]\). In particular, \( L(c) \geq L(0) = (\sum_{i=1}^{n} p_i T_i) e \), which is the Ahmad lower bound for \( M_2 \) given in Theorem B. From (17), we have

\[ w(x) \leq w(c) + D(x - c) \equiv H_x(c). \]

Differentiating \( H_x(c) \) with respect to \( c \) gives, for any \( x \) in \([x^*, x^{**}]\),

\[
\begin{align*}
\frac{d}{dc} H_x(c) &= w'(c) - D = w'(c) - w'(x^{**}) \leq 0,
\end{align*}
\]

since \( c \leq x^{**} \) and \( w'' \geq 0 \) on \((0, \infty)\). Thus \( H_x(c) \) is nonincreasing in \( c \) for each \( x \) with \([x^*, x^{**}]\) which gives, upon exponentiation, that \( U(c) \) is nonincreasing on \([0, x^*]\). This proves (d).

**Remark 2.5** It is conjectured that \( U(x^*) \leq U_A \). Not one case where \( U(x^*) \geq U_A \) has been found in many numerical comparisons done, some of which is given in Section 3 later. In any case, Theorem 3 presented next gives a new upper bound, \( U_2 \), which satisfies \( U_2 \leq U_A \) always. (It was noticed that \( U(x^*) \leq U_2 \) as well, but this could not be proven either.) Thus, \( L(x^*) \) and \( U_2 \) improve upon \( L_A \) and \( U_A \), respectively.

Next, we present another upper bound of \( M_2 \) which sometimes is better than \( U(c) \) and which is at least as good as \( U_A \), the Ahmad upper bound of Theorem B.

**Theorem 2.6** We have \( M_2 \leq (pe + q) \equiv U_2 \), where

\[
p = \sum_{i=1}^{n} p_i T_i \quad \text{and} \quad q = \sum_{i=1}^{n} p_i (T_{\max} - T_i).
\]

Also, \( U_2 \leq U_A \).
Proof. In Ahmad [2004], it is established that
\[ M_2 \leq \min_{y>0} \frac{pe^y + q}{y}. \] (18)
By Theorem 1, part (a) with the above choices of \( p \) and \( q \) and \( T = 1 \), the result \( M_2 \leq U_2 \) follows immediately. The second part follows immediately from Theorem 1, part (c) with the above choices of \( p, q \) and \( T \), since \( U_A \) is also based on an application of (18) above. See Ahmad [2004], p. 4.

Remark 2.7 We have established that \( L_A \leq L(c) \) and \( U_2 \leq U_A \) in Theorems 2 and 3, improving on both bounds of Theorem B. The best lower bound \( L(c) \) value occurs when \( c = x^* \).

Next, we present a very accurate approximation of \( M_2 \), but necessarily bounds for \( M_2 \). The bounds \( L(c) \) and \( U(c) \) were found by taking natural logarithms of \( g(x) = \sum_{i=1}^{n} p_i e^{x_i} \), that is \( w(x) = \ln(g(x)) \) bounds were found. Here, we use instead \( w(x) = g(x)^\delta \) for a suitable \( \delta < 0 \). How to choose \( \delta \)? One way is so that \( w(x) = (g(x))^\delta \) is very nearly linear. This requires \( w''(x) = 0 \).

After some algebra, this produces the choice \( \delta \) satisfying
\[ g(x)g''(x) + (g'(x))^2(\delta - 1) = 0. \] (19)
Solving (19) for \( \delta \), we obtain
\[ \delta = 1 - \frac{g(x)g''(x)}{(g'(x))^2} \leq 0. \] (20)
(If \( \delta = 0 \), a L'Hospital's Rule type of argument applied here leads to \( w(x) = \ln(g(x)) \). It can be shown \( \delta = 0 \) if and only if \( g(x) = ae^{bx} \).) We shall use (20) for \( x = x^* \), since \( L(x^*) \) is usually substantially closer to \( M_2 \) than is \( U(x^*) \). This will be seen later in numerical comparisons. Thus, we consider the choice
\[ \delta = \delta^* = 1 - \frac{g(x^*)g''(x^*)}{(g'(x^*))^2}. \] (21)
Then we obtain: the approximation (not bounds)
\[ w(x) \approx [g(x^*)^\delta - \delta x^*(g(x^*))^{\delta-1}g'(x^*)] \]
\[ + [\delta(g(x^*))^{\delta-1}g'(x^*)] \cdot x \equiv A^* + B^*x \]
which gives
\[ g(x) \approx (A^* + B^*x)^{\frac{1}{\delta}}. \]
So
\[ M_2 = \min_{x>0} f(x) = \min_{x>0} \frac{g(x)}{x} \approx \min_{x>0} \frac{(A^* + B^*x)^{\frac{1}{\delta}}}{x}. \]
Some new inequalities

The latter minimum value occurs at $x = \frac{A^*}{B^*(\frac{1}{2} - 1)} \equiv X_*$. Corresponding minimum value $\left(\frac{A^*}{1 - \delta}\right)^{\frac{1}{2}}$. Thus, the approximation to $M_2$ is

$$M_2^* = \frac{\left(\frac{A^*}{1 - \delta}\right)^{\frac{1}{2}}}{X_*},$$

(22)

evaluated at $\delta = \delta^*$.

3 Numerical Comparisons

In this section, we compare the bounds ($L_A$ and $U_A$ of Theorem B) of Ahmad (2004) to the new bounds ($L(x^*)$, $U(x^*)$ and $U_2$) presented in this paper. We present the bounds for

$$M_2 = \min_{x > 0} \frac{\sum_{i=1}^{n} p_i e^{T_i x}}{x}$$

for $n = 2$ first. This choice of $n$ was used by Ahmad (2004) also. The first three rows of Table 1 below correspond to choices for $p_1$, $p_2$, $T_1$, and $T_2$ used by Ahmad (2004) in his Section 3 table. We also present the value of the approximation $M_2^*$ (but not necessarily a bound) discussed in Section 2.

Table 1

<table>
<thead>
<tr>
<th>$p_1$</th>
<th>$p_2$</th>
<th>$T_1$</th>
<th>$T_2$</th>
<th>$L_A$</th>
<th>$L(x^*)$</th>
<th>$M_2$</th>
<th>$M_2^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0.3</td>
<td>9.7858</td>
<td>10.6696</td>
<td>10.6885</td>
<td>10.6891</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0.9</td>
<td>13.0478</td>
<td>13.0645</td>
<td>13.0646</td>
<td>13.0646</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0.98</td>
<td>13.4827</td>
<td>13.4833</td>
<td>13.4833</td>
<td>13.4833</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0.05</td>
<td>8.4267</td>
<td>10.0357</td>
<td>10.0700</td>
<td>10.0718</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>1</td>
<td>0.20</td>
<td>5.3466</td>
<td>6.8530</td>
<td>7.3415</td>
<td>7.3759</td>
</tr>
<tr>
<td>10</td>
<td>2</td>
<td>1</td>
<td>0.50</td>
<td>29.9011</td>
<td>30.4340</td>
<td>30.4352</td>
<td>30.4351</td>
</tr>
</tbody>
</table>

Table 2

<table>
<thead>
<tr>
<th>$p_1$</th>
<th>$p_2$</th>
<th>$T_1$</th>
<th>$T_2$</th>
<th>$M_2$</th>
<th>$U_A$</th>
<th>$U(x^*)$</th>
<th>$U_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0.3</td>
<td>10.6885</td>
<td>11.2909</td>
<td>10.7360</td>
<td>11.1858</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0.9</td>
<td>13.0646</td>
<td>13.2493</td>
<td>13.0646</td>
<td>13.2478</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0.98</td>
<td>13.4833</td>
<td>13.5227</td>
<td>13.4833</td>
<td>13.5227</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0.05</td>
<td>10.0700</td>
<td>10.5579</td>
<td>10.1779</td>
<td>10.3267</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>1</td>
<td>0.20</td>
<td>7.3415</td>
<td>11.3465</td>
<td>8.5000</td>
<td>9.4366</td>
</tr>
<tr>
<td>10</td>
<td>2</td>
<td>1</td>
<td>0.50</td>
<td>30.4340</td>
<td>30.9180</td>
<td>30.4377</td>
<td>30.9011</td>
</tr>
</tbody>
</table>
From Tables 1 and 2, we see that the new lower and upper bounds greatly improve on the bounds of Ahmad (2004) when $T_1$ and $T_2$ differ greatly. Only when $T_1$ and $T_2$ are nearly equal are the new bounds only slightly better. We see that the approximation $M_2$ is a very good one also in most cases. For $n \geq 3$, we present a few cases in Tables 3 and 4 below.

**Case A:** $n = 3, p_1 = 2, T_1 = 10, p_2 = 1, T_2 = 1, p_3 = 5, T_3 = 0.10$

**Case B:** $n = 3, p_1 = 1, T_1 = 5, p_2 = 5, T_2 = 0.2, p_3 = 4, T_3 = 20.0$

**Case C:** $n = 3, p_1 = 10, T_1 = 2, p_2 = 3, T_2 = 3, p_3 = 0.10, T_3 = 10.0$

### Table 3
(Lower Bounds for $M_2$ for Cases A, B, C and $M_2^*$)

<table>
<thead>
<tr>
<th>Case</th>
<th>$L_A$</th>
<th>$L(x^*)$</th>
<th>$M_2$</th>
<th>$M_2^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>58.4431</td>
<td>93.8818</td>
<td>101.0014</td>
<td>101.9014</td>
</tr>
<tr>
<td>B</td>
<td>233.772</td>
<td>318.030</td>
<td>324.363</td>
<td>324.912</td>
</tr>
<tr>
<td>C</td>
<td>81.548</td>
<td>84.222</td>
<td>91.306</td>
<td>88.902</td>
</tr>
</tbody>
</table>

### Table 4
(Upper Bounds for $M_2$ for Cases A, B, C)

<table>
<thead>
<tr>
<th>Case</th>
<th>$M_2$</th>
<th>$U_A$</th>
<th>$U(x^*)$</th>
<th>$U_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>101.0014</td>
<td>159.0196</td>
<td>115.6028</td>
<td>116.9431</td>
</tr>
<tr>
<td>B</td>
<td>324.363</td>
<td>380.696</td>
<td>340.771</td>
<td>347.772</td>
</tr>
<tr>
<td>C</td>
<td>91.306</td>
<td>281.385</td>
<td>115.390</td>
<td>182.548</td>
</tr>
</tbody>
</table>

Again, we see that the new bounds are substantially better when the $T_i$ values are quite diverse. Also, $M_2^*$ remains a very good approximation of $M_2$. These conclusions remain the same for $n \geq 4$.

The new bounds presented in this paper can be extended to bounds for $f(x) = \frac{g(x)}{x}$, where $g(x)$ is absolutely monotonic on $(0, \infty)$ having form (6).
References


Received: July 21, 2015; Published: August 25, 2015