Inequalities for Selfadjoint Operators on Hilbert Spaces and Pseudo-Hilbert Spaces

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Abstract

In this paper are given some inequalities for selfadjoint operators on Hilbert spaces and pseudo-Hilbert spaces starting from an inequality of Kober and from a refinement of the Kittaneh-Manasrah inequality by combining inequalities for power series and inner product.

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1. Introduction

The famous inequality of Young

\[ \lambda a + (1 - \lambda)b \geq a^{\lambda}b^{1-\lambda} \]

where \(a, b\) are positive real numbers and \(\lambda \in [0, 1]\) has many improvements and refinements.

We need below the following one, the improvement of the inequality between arithmetic and geometric means for \(n = 2\) given by H. Kober in [9]:

\[ r(\sqrt{a} - \sqrt{b})^2 \leq \lambda a + (1 - \lambda)b - a^{\lambda}b^{1-\lambda} \leq (1 - r)(\sqrt{a} - \sqrt{b})^2 \]

where \(a, b\) are positive real numbers, \(\lambda \in [0, 1]\) and \(r = \min\{\lambda, 1 - \lambda\} \).

We use also below the following result in the next section.
Proposition([10]) For $0 < a, b \leq 1$ and $\lambda \in (0, 1)$ we have
\[
\begin{align*}
    r(\sqrt{a} - \sqrt{b})^2 + A(\lambda)ab \log^2 \left(\frac{a}{b}\right) & \leq \lambda a + (1 - \lambda)b - a^\lambda b^{1-\lambda} \\
    \leq (1 - r)(\sqrt{a} - \sqrt{b})^2 + B(\lambda)ab \log^2 \left(\frac{a}{b}\right)
\end{align*}
\]
where $r = \min\{\lambda, 1 - \lambda\}$, $A(\lambda) = \frac{\lambda(1 - \lambda)}{2} - \frac{r}{2}$ and $B(\lambda) = \frac{\lambda(1 - \lambda)}{2} - \frac{1 - r}{4}$.

We will take here $\lambda = \frac{1}{p}$ and replace $a^\lambda$ by $a$ and $b^{1-\lambda}$ by $b$ then $1 - \lambda = \frac{1}{q}$ and we obtain:
\[
\begin{align*}
    ab + r(a^{\frac{1}{p}} - b^{\frac{1}{q}})^2 + A\left(\frac{1}{p}\right)a^p b^q \log^2 \left(\frac{a^p}{b^q}\right) & \leq \frac{ap}{p} + \frac{bq}{q} \\
    \leq ab + (1 - r)(a^{\frac{1}{p}} - b^{\frac{1}{q}})^2 + B\left(\frac{1}{p}\right)a^p b^q \log^2 \left(\frac{a^p}{b^q}\right).
\end{align*}
\]
(1.1)

We consider here an analytic function defined by the power series
\[
f(z) = \sum_{n=0}^{\infty} a_n z^n
\]
with real coefficients and convergent on the disk $D(0, R)$, $R > 0$. As in [8] the weighted version of Hölder’s inequality can be stated as below:
\[
|f(xy)| = \left|\sum_{n=0}^{\infty} a_n x^n y^n\right| \leq \left(\sum_{n=0}^{\infty} |a_n||x|^{pn}\right)^\frac{1}{p} \left(\sum_{n=0}^{\infty} |a_n||y|^{qn}\right)^\frac{1}{q}
\]
\[
= f_A^\frac{1}{p}(|x|^p)f_A^\frac{1}{q}(|y|^q)
\]
for any $x, y \in \mathbb{C}$ with $xy$, $|x|^p$, $|y|^q \in D(0, R)$ and $f_A(z)$ is a power series defined by $\sum_{n=0}^{\infty} |a_n|z^n$. The power series $f_A(z)$ have the same radius of convergence as the original power series $f(z)$.

In the case when all coefficients of the series $f(z)$ are positive we have $f(z) = f_A(z)$.

First, it is necessary to recall that for selfadjoint operators $A, B \in B(H)$ we write $A \leq B$ (or $B \geq A$) if $< Ax, x > \leq < Bx, x >$ for every vector $x \in H$. We will consider for beginning $A$ as being a selfadjoint linear operator on a complex Hilbert space $(H; < \cdot, \cdot >)$ as in [7] and the references therein. The Gelfand map establishes a *-isometrically isomorphism $\Phi$ between the set $C(Sp(A))$ of all continuous functions defined on the spectrum of $A$, denoted $Sp(A)$, and the $C^*$- algebra $C^*(A)$ generated by $A$ and the identity operator $1_H$ on $H$ as follows: For any $f, g \in C(Sp(A))$ and for any $\alpha, \beta \in \mathbb{C}$ we have
\[
\begin{align*}
    (i) \quad & \Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g) \\
    (ii) \quad & \Phi(fg) = \Phi(f)\Phi(g) \quad \text{and} \quad \Phi(f) = \Phi(f^*) \\
    (iii) \quad & ||\Phi(f)|| = ||f|| := \sup_{t \in Sp(A)} |f(t)|
\end{align*}
\]
(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$ for $t \in Sp(A)$.

Using this notation, as in [7] for example, we define

$$f(A) := \Phi(f) \quad \text{for all} \quad f \in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator $A$. It is known that if $A$ is a selfadjoint operator and $f$ is a real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e. $f(A)$ is a positive operator on $H$. In addition, if and $f$ and $g$ are real valued functions on $Sp(A)$ then the following property holds:

$$(1.2) \quad f(t) \geq g(t) \quad \text{for any} \quad t \in Sp(A) \quad \text{implies that} \quad f(A) \geq g(A)$$

in the operator order of $B(H)$.

We recall the definitions of pseudo-Hilbert spaces (so called Loeynes $Z$- spaces) and of the admissible spaces in the Loeynes sense and then as in [2] we use the functional calculus with functions of the class $C_1$ in results which will be proved below.

A locally convex space $Z$- is called admissible in the Loeynes sense if the following conditions are satisfied:

(A.1) $Z$- is complete;

(A.2) there is a closed convex cone in $Z$, denoted $Z_+$, defines an order relation on $Z$ (that is $z_1 \leq z_2$ if $z_2 - z_1 \in Z_+$);

(A.3) there is an involution in $Z$, $z \mapsto z^*$ (that is $z^{**} = z$, $(az)^* = \overline{a} z^*$, $(z_1 + z_2)^* = z_1^* + z_2^*$) such that $z \in Z_+$ implies $z^* = z$;

(A.4) the topology of $Z$ is compatible with the order (that is there exists a basis of convex solid neighbourhoods of the origin);

(A.6) any monotonously decreasing sequence in $Z_+$ is convergent.

Let $Z$ be an admissible space in the Loeynes sense. A topological linear space $H$ is called pre-Loynes $Z$-space if it satisfies the following properties:

(L1) $H$ is endowed with an $Z$- valued inner product (gramian), i.e. there exists an application $(h,k) \in H \times H \rightarrow [h,k] \in Z$ having the properties:

(G.1) $[h,h] \geq 0$; $[h,h] = 0$ implies $h = 0$;

(G.2) $[h_1 + h_2, h] = [h_1, h] + [h_2, h]$;

(G.3) $[\lambda h, k] = \lambda[h,k]$;

(G.4) $[h,k]^* = [k,h]$;

for all $h, k, h_1, h_2 \in H$ and $\lambda \in \mathbb{C}$.

(L.2) The topology of $H$ is the weakest locally convex topology on $H$ for which the application $h \in H \rightarrow [h,h] \in Z$ is continuous.

Moreover, if $H$ is a complete spaces with this topology, then $H$ is called Loeynes $Z$- space (pseudo-Hilbert space).

Let $A \in B^*_h(H)$, where $H$ is now a pseudo-Hilbert space (so called Loeynes $Z$-spaces) and

$$m_A := \sup\{\mu : \mu[h,h] \leq [Ah,h], h \in H\},$$

$$M_A := \inf\{\nu : [Ah,h] \leq \nu[h,h], h \in H\}.$$
We say that the function $f$ is in the class $C_1$ and we denote $f \in C_1[m_A, M_A]$ if $f$ is positive and superior semicontinuous on $[m_A, M_A]$.

We will denote by $\overline{\mathcal{A}(A)^f}$ the strong closing of $\mathcal{A}(A)$ in $B^+(\mathcal{H})$.

The mapping

$$f : C_1[m_A, M_A] \to \overline{\mathcal{A}(A)^f}, \quad f \to f(A)$$

by which to a function $f \in C_1[m_A, M_A]$ we associate the gramian self-adjoint operator denoted by $f(A)$ and defined by $f(A) = \lim_{n \to \infty} p_n(A)$ where $p_n$ is a decreasing sequence of polynomials $p_n$ with $f(\lambda) = \lim_{n \to \infty} p_n(\lambda)$ for any $\lambda \in [m_A, M_A]$, is called functionals calculus with functions in class $C_1$.

**Theorem 1.** (Lemma 2.1.1[2]) The functional calculus with functions of the class $C_1$ has the following immediate properties:

(i) the mapping $f \to f(A)$ is monotone;

(ii) $f \to f(A)$ is function of positive type and positively homogeneous;

(iii) $f \to f(A)$ is additive and multiplicative (all these three properties being inherited by passing to the limit from the functional calculus with polynomials defined at the beginning);

(iv) In addition, the functional calculus with functions of the class $C_1$ extends the functional calculus with continuous and positive functions on $\sigma(A)$ defined in Corollary 1.5.6,

$$f : C_+(\sigma(A)) \to \mathcal{A}(A), \quad f \to f(A)$$

if $A \in \mathcal{B}_1^+(\mathcal{H})$.

Using the definition from [7], we say that the functions $f, g : [a, b] \to \mathbb{R}$ are synchronous (asynchronous) on the interval $[a, b]$ if they satisfy the following condition:

$$(f(t) - f(s))(g(t) - g(s)) \geq (\leq) 0$$

for each $t, s \in [a, b]$.

Some of inequalities from this paper are continuation of some inequalities from previous papers as [3], [4], [5] and [6].

2. Some inequalities for selfadjoint operators on Hilbert spaces and on pseudo-Hilbert spaces

Some refinements of Young inequalities for power series and inner product and positive operators on Hilbert spaces and pseudo-Hilbert spaces will be presented in the following theorems.

First result was established starting from the inequality (1.0) and following the same reason as in [8] and [7].

**Theorem 2.** Let $f(z) = \sum_{n=0}^{\infty} p_n z^n$, $g(z) = \sum_{n=0}^{\infty} q_n z^n$ be the power series with real coefficients and convergent on the open disk $D(0, R)$, $0 < R < 1$ (or $R > 0$). If $p, q$ are real numbers with $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $A_1$ is a positive definite operator on the
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Hilbert space $\mathcal{H}$, $A_1 \in B(H)$ with $Sp(A_1) \subseteq (0, R)$ and $B$ is a positive operator on the pseudo-Hilbert space $\mathcal{K}$, $B \in B^*_r(\mathcal{K})$ with $Sp(B) \subseteq (0, R)$, then we have

$$
< f_A \left( A_1^{\frac{p}{q}} \right)^x, x > [g_A \left( B_1^{\frac{p}{q}} \right) y, y] + \sum_{j} \sum_{k} < f_A \left( A_1^{\frac{p}{q}} \right) x, x > [g_A \left( B_1^{\frac{p}{q}} \right) y, y] + < f_A \left( A_1^{\frac{p}{q}} \right) x, x > [g_A \left( B_1^{\frac{p}{q}} \right) y, y] \\
\leq \frac{< f_A \left( A_1^{\frac{p}{q}} \right) x, x > [g_A \left( B_1^{\frac{p}{q}} \right) y, y]}{q} + \frac{< f_A \left( A_1^{\frac{p}{q}} \right) x, x > [g_A \left( B_1^{\frac{p}{q}} \right) y, y]}{p} \\
\leq < f_A \left( A_1^{\frac{p}{q}} \right) x, x > [g_A \left( B_1^{\frac{p}{q}} \right) y, y] + (1 - r) \{ < f_A \left( A_1^{\frac{p}{q}} \right) x, x > [g_A \left( B_1^{\frac{p}{q}} \right) y, y] + < f_A \left( A_1^{\frac{p}{q}} \right) x, x > [g_A \left( B_1^{\frac{p}{q}} \right) y, y] \}
$$

for each $x \in H$ and $y \in \mathcal{K}$, where $\lambda \in [0, 1]$, $r = \min \{ \lambda, 1 - \lambda \}$.

Proof. As in [8], we take in inequality (1.1) $|a|^{k \frac{p}{q}} |b|^j$ instead of $a$ and $|a|^{k \frac{p}{q}} |b|^j$ instead of $b$ where $k, j \in \mathbb{N}$, we do not take into account the terms $A(\frac{1}{p})a^b \log^2 \left( \frac{a^b}{br} \right)$ and $B(\frac{1}{p})a^b \log^2 \left( \frac{a^b}{br} \right)$ (that is a form of inequality (1.0)) and we obtain:

$$
|a|^{k \frac{p}{q}} |b|^j \leq |a|^{k \frac{p}{q}} |b|^j + |a|^{k \frac{p}{q}} |b|^j - 2|a|^{k \frac{p}{q}} |b|^j \leq \frac{|a|^{k \frac{p}{q}} |b|^j}{p} + \frac{|a|^{k \frac{p}{q}} |b|^j}{q} \\
\leq |a|^{k \frac{p}{q}} |b|^j + (1 - r) \left( |a|^{k \frac{p}{q}} |b|^j + |a|^{k \frac{p}{q}} |b|^j - 2|a|^{k \frac{p}{q}} |b|^j \right).
$$

If we multiply the inequality with positive quantities $|p_j||q_k|$ and sum over $j$ and $k$ from 0 to $n$, we have

$$
\sum_{k=0}^{n} \sum_{j=0}^{n} |p_j||q_k||a|^{k \frac{p}{q}} |b|^j \leq \sum_{k=0}^{n} \sum_{j=0}^{n} \left( |a|^{k \frac{p}{q}} |b|^j + |a|^{k \frac{p}{q}} |b|^j - 2|a|^{k \frac{p}{q}} |b|^j \right) \\
\leq \sum_{k=0}^{n} \sum_{j=0}^{n} \frac{|a|^{k \frac{p}{q}} |b|^j}{p} + \sum_{k=0}^{n} \sum_{j=0}^{n} \frac{|a|^{k \frac{p}{q}} |b|^j}{q} \leq \sum_{k=0}^{n} \sum_{j=0}^{n} |p_j||q_k||a|^{k \frac{p}{q}} |b|^j + (1 - r) \sum_{k=0}^{n} \sum_{j=0}^{n} |p_j||q_k||a|^{k \frac{p}{q}} |b|^j
$$

All the series whose partial sums which appear in previous inequality are convergent on the disk $D(0, R)$ therefore we can take the limit when $n$ tends to $\infty$ before and obtain the inequality

$$
f_A(|a|^\frac{p}{q})g_A(|b|^\frac{p}{q}) + r \{ f_A(|a|^p)g_A(|b|^p) + f_A(|a|^q)g_A(|b|^q) - 2f_A(|a|^\frac{p}{q})g_A(|b|^\frac{p}{q}) \} \leq \frac{r}{p} f_A(|a|^p)g_A(|b|^p) + \frac{r}{q} f_A(|a|^q)g_A(|b|^q) \\
\leq f_A(|a|^\frac{p}{q})g_A(|b|^\frac{p}{q}) + (1 - r) \{ f_A(|a|^p)g_A(|b|^p) + f_A(|a|^q)g_A(|b|^q) - 2f_A(|a|^\frac{p}{q})g_A(|b|^\frac{p}{q}) \}.
$$
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Now we fix $|a|$ and apply poperty (1.2) obtaining for any $x \in \mathcal{H}$,

\[
< f_A(|A_1|^q)x, x > g_A(|b|^q) + r\{ < f_A(|A_1|^p)x, x > g_A(|b|^p) + < f_A(|A_1|^q)x, x > g_A(|b|^q) - 2 < f_A(|A_1|^q)x, x > g_A(|b|^q) \} \leq \frac{< f_A(|A_1|^p)x, x > g_A(|b|^p)}{p} + \frac{< f_A(|A_1|^q)x, x > g_A(|b|^q)}{q} \leq < f_A(|A_1|^q)x, x > g_A(|b|^q) + (1 - r)\{ < f_A(|A_1|^p)x, x > g_A(|b|^p) + 2 < f_A(|A_1|^q)x, x > g_A(|b|^q) \}.
\]

Applying now for $|b|$ fixed the Theorem 1, for any $y \in \mathcal{K}$ we get the desired inequality.

\[\square\]

The second result was given starting from inequality (1.1), see [7] and following the same reason as in [8] and [7].

**Theorem 3.** Let \( f(z) = \sum_{n=0}^{\infty} p_n z^n \), \( g(z) = \sum_{n=0}^{\infty} q_n z^n \) be the power series with real coefficients and convergent on the open disk \( D(0, R) \), \( 0 < R < 1 \). If \( p, q \) are real numbers with \( p > 1, \frac{1}{p} + \frac{1}{q} = 1 \) and \( A_1 \) is a positive definite operator on the Hilbert space \( \mathcal{H} \), \( A_1 \in B(\mathcal{H}) \) with \( Sp(A_1) \subseteq (0, R) \) and \( B \) is a positive operator on the pseudo-Hilbert space \( \mathcal{K} \), \( B \in B_{\mathcal{K}}(\mathcal{K}) \) with \( Sp(B) \subseteq (0, R) \) then we have

\[
< f_A\left(A_1^{\frac{q}{p}+1}\right)x, x > [g_A\left(B^{\frac{q}{p}+1}\right)y, y] + r\{ < f_A\left(A_1^q\right)x, x > [g_A\left(B^p\right)y, y] + \}
\]

\[
+ < f_A\left(A_1^q\right)x, x > [g_A\left(B^2\right)y, y] - 2 < f_A\left(A_1^{q+1}\right)x, x > [g_A\left(B^{q+1}\right)y, y]\}
\]

\[
\leq \frac{< f_A\left(A_1^q\right)x, x > [g_A\left(B^p\right)y, y]}{p} + \frac{< f_A\left(A_1^q\right)x, x > [g_A\left(B^2\right)y, y]}{q} \leq < f_A\left(A_1^{\frac{q}{p}+1}\right)x, x > [g_A\left(B^{\frac{q}{p}+1}\right)y, y] + (1 - r)\{ < f_A\left(A_1^q\right)x, x > [g_A\left(B^p\right)y, y] + \}
\]

\[
+ < f_A\left(A_1^q\right)x, x > [g_A\left(B^2\right)y, y] - 2 < f_A\left(A_1^{q+1}\right)x, x > [g_A\left(B^{q+1}\right)y, y] \} + B_2\left(\frac{1}{p}\right) M, \]

where

\[
M = (2 - q)^2 < S_1\left|A_1^{2+q}\right| \log^2(A_1)x, x > [g\left(B^{p+2}\right)y, y] + \]

\[
+ (p - 2)^2 < f\left|A_1^{2+q}\right)x, x > [S_2\left(B^{p+2}\right) \log^2(B)y, y] + \]

\[
+ 2(2 - q)(p - 2) < S_3\left|A_1^{2+q}\right| \log(A_1)x, x > [S_4\left(B^{p+2}\right) \log(B)y, y], \]

and \( S_1(a) = af_A'(a) + a^2 f_A''(a) \), \( S_2(a) = ag_A'(a) + a^2 g_A''(a) \), \( S_3(a) = af_A'(a) \), \( S_4(a) = ag_A'(a) \) for each \( x \in \mathcal{H} \) and \( y \in \mathcal{K} \), where \( \lambda \in [0, 1], r = \min\{\lambda, 1 - \lambda\} \).
Proof. Using the same reason as in previous theorem, we take in inequality (1.1) $|a|^{k\frac{2}{p}+1}|b|^j(\frac{2}{q}+1)$ instead of $a$ and $|a|^k|b|^j\frac{2}{q}$ instead of $b$ where $k, j \in \mathbb{N}$ and we obtain:

$$|a|^{k\frac{2}{p}+1}|b|^j(\frac{2}{q}+1) + r\left(|a|^{2k}|b|^{jp} + |a|^q|b|^{2j} - 2|a|^{k+\frac{2}{q}}|b|^{j(1+\frac{2}{q})}\right) +$$

$$+ A_2 \left(\frac{1}{p}\right) |a|^{2k}|b|^{jp}|a|^q|b|^{2j} \log^2 \left(\frac{|a|^{2k}|b|^{jp}}{|a|^q|b|^{2j}}\right) \leq \frac{|a|^{2k}|b|^{jp}}{p} + \frac{|a|^q|b|^{2j}}{q} \leq$$

$$\leq |a|^{k\frac{2}{p}+1}|b|^j(\frac{2}{q}+1) + (1 - r) \left(|a|^{2k}|b|^{jp} + |a|^q|b|^{2j} - 2|a|^{k+\frac{2}{q}}|b|^{j(1+\frac{2}{q})}\right) +$$

$$+ B_2 \left(\frac{1}{p}\right)|a|^{2k}|b|^{jp}|a|^q|b|^{2j} \log^2 \left(\frac{|a|^{2k}|b|^{jp}}{|a|^q|b|^{2j}}\right).$$

If we multiply the inequality with positive quantities $|p_j|q_k$ and sum over $j$ and $k$ from 0 to $n$, we have:

$$\sum_{k=0}^{n} \sum_{j=0}^{n} |p_j||q_k||a|^{k\frac{2}{p}+1}|b|^j(\frac{2}{q}+1) + r\sum_{k=0}^{n} \sum_{j=0}^{n} |p_j||q_k| \left(|a|^{2k}|b|^{jp} + |a|^q|b|^{2j} - 2|a|^{k+\frac{2}{q}}|b|^{j(1+\frac{2}{q})}\right) +$$

$$+ A_2 \left(\frac{1}{p}\right) \sum_{k=0}^{n} \sum_{j=0}^{n} |p_j||q_k| \left(|a|^{2k}|b|^{jp}|a|^q|b|^{2j} \log^2 \left(\frac{|a|^{2k}|b|^{jp}}{|a|^q|b|^{2j}}\right) \right) \leq$$

$$\leq \sum_{k=0}^{n} \sum_{j=0}^{n} |p_j||q_k| |a|^{k\frac{2}{p}+1}|b|^j(\frac{2}{q}+1) +$$

$$+(1 - r) \sum_{k=0}^{n} \sum_{j=0}^{n} |p_j||q_k| \left(|a|^{2k}|b|^{jp} + |a|^q|b|^{2j} - 2|a|^{k+\frac{2}{q}}|b|^{j(1+\frac{2}{q})}\right) +$$

$$+ B_2 \left(\frac{1}{p}\right) \sum_{k=0}^{n} \sum_{j=0}^{n} |p_j||q_k| \left(|a|^{2k}|b|^{jp}|a|^q|b|^{2j} \log^2 \left(\frac{|a|^{2k}|b|^{jp}}{|a|^q|b|^{2j}}\right) \right),$$

or by calculus,

$$\sum_{k=0}^{n} \sum_{j=0}^{n} |p_j||q_k| |a|^{k\frac{2}{p}+1}|b|^j(\frac{2}{q}+1) +$$

$$+ r\sum_{k=0}^{n} \sum_{j=0}^{n} |p_j||q_k| \left(|a|^{2k}|b|^{jp} + |a|^q|b|^{2j} - 2|a|^{k+\frac{2}{q}}|b|^{j(1+\frac{2}{q})}\right) +$$

$$+ A_2 \left(\frac{1}{p}\right) \sum_{k=0}^{n} \sum_{j=0}^{n} |p_j||q_k| |a|^{k\frac{2}{p}+1}|b|^j(\frac{2}{q}+1) +$$

$$+ 2k(j(2-q)(p-2) \log(|a|)) \leq \sum_{k=0}^{n} \sum_{j=0}^{n} |p_j||q_k| \frac{|a|^{2k}|b|^{jp}}{p} + \sum_{k=0}^{n} |p_j||q_k| \sum_{j=0}^{n} \frac{|a|^q|b|^{2j}}{q} \leq$$

$$\leq \sum_{k=0}^{n} \sum_{j=0}^{n} |p_j||q_k| |a|^{k\frac{2}{p}+1}|b|^j(\frac{2}{q}+1) +$$
\begin{align*}
+(1 - r) & \sum_{k=0}^{n} \sum_{j=0}^{n} |p_j||q_k| \left( |a|^{2k}|b|^{jp} + |a|^{q_k}|b|^{2q_j} - 2|a|^{k\left(1 + \frac{2}{q}\right)}|b|^{j\left(1 + \frac{2}{q}\right)} \right) + \\
+ B_2 \left( \frac{1}{p} \right) & \sum_{k=0}^{n} \sum_{j=0}^{n} |p_j||q_k||a|^{k(q+2)}|b|^{j(p+2)}(k^2(2 - q)^2 \log^2(|a|) + j^2(p - 2)^2 \log^2(|b|)) \\
& + 2kj(2 - q)(p - 2) \log(|a|).
\end{align*}

All the series whose partial sums which appear in previous inequality are convergent on the disk \(D(0, R)\) therefore we can take the limit when \(n\) tends to \(\infty\) before and obtain the inequality

\[ f_A(|a|^{\frac{q}{2}+1})g_A(|b|^{\frac{q}{2}+1}) + r \left\{ f_A(|a|^2)g_A(|b|^p) + f_A(|a|^q)g_A(|b|^2) - 2f_A(|a|^{\frac{q}{2}+1})g_A(|b|^{\frac{q}{2}+1}) \right\} + \]

\[ + A_2 \left( \frac{1}{p} \right) \left\{ (2 - q)^2 \log^2(|a|)g(|b|^{p+2}) \left[ |a|^{2+q}|f'(|a|^{2+q})| + |a|^{2+q}f''(|a|^{2+q}) \right] + \right. \]

\[ + (p - 2)^2 \log^2(|b|)f(|a|^{2+q}) \left[ |b|^{p+2}|g'(|b|^{p+2})| + |b|^{2(p+2)}g''(|b|^{p+2}) \right] + \]

\[ + 2(2 - q)(p - 2) \log(|a|) \log(|b|) \left[ |a|^{2+q}f'(|a|^{2+q})|b|^{p+2}g'(|b|^{p+2}) \right] \leq \]

\[ \leq \frac{f_A(|a|^2)g_A(|b|^p)}{p} + \frac{f_A(|a|^q)g_A(|b|^2)}{q} \leq \]

\[ \leq f_A(|a|^{\frac{q}{2}+1})g_A(|b|^{\frac{q}{2}+1}) + (1 - r) \left\{ f_A(|a|^2)g_A(|b|^p) + f_A(|a|^q)g_A(|b|^2) - 2f_A(|a|^{\frac{q}{2}+1})g_A(|b|^{\frac{q}{2}+1}) \right\} + \]

\[ + B_2 \left( \frac{1}{p} \right) \left\{ (2 - q)^2 \log^2(|a|)g(|b|^{p+2}) \left[ |a|^{2+q}|f'(|a|^{2+q})| + |a|^{2+q}f''(|a|^{2+q}) \right] + \right. \]

\[ + (p - 2)^2 \log^2(|b|)f(|a|^{2+q}) \left[ |b|^{p+2}|g'(|b|^{p+2})| + |b|^{2(p+2)}g''(|b|^{p+2}) \right] + \]

\[ + 2(2 - q)(p - 2) \log(|a|) \log(|b|) \left[ |a|^{2+q}f'(|a|^{2+q})|b|^{p+2}g'(|b|^{p+2}) \right] \leq \]

Now we fix \(|a|\) and apply property (1.2) obtaining for any \(x \in \mathcal{H},\)

\[ < f_A(A_1^{\frac{q}{2}+1})x, x > > g_A(|b|^{\frac{q}{2}+1}) + r \left\{ < f_A(A_1^q)x, x > > g_A(|b|^p) + \right. \]

\[ + < f_A(A_1^q)x, x > > g_A(|b|^2) - 2 < f_A(A_1^{\frac{q}{2}+1})x, x > > g(|b|^{\frac{q}{2}+1}) \right\} + \]

\[ + A_2 \left( \frac{1}{p} \right) \left\{ (2 - q)^2 < S_1(A_1^{2+q}) \log^2(A_1)x, x > > g(|b|^{p+2}) + \right. \]

\[ + (p - 2)^2 < f(A_1^{2+q})x, x > > S_2(|b|^{p+2}) \log^2(|b|) \right\} + \]

\[ + 2(2 - q)(p - 2) < S_3(A_1^{2+q}) \log(A_1)x, x > > S_4(|b|^{p+2}) \log(|b|) \leq \]

\[ \leq \frac{< f_A(A_1^q)x, x > > g_A(|b|^p)}{p} + \frac{< f_A(A_1^q)x, x > > g_A(|b|^2)}{q} \leq \]

\[ \leq < f_A(A_1^{\frac{q}{2}+1})x, x > > g_A(|b|^{\frac{q}{2}+1}) + (1 - r) \left\{ < f_A(A_1^q)x, x > > g_A(|b|^p) + \right. \]

\[ + < f_A(A_1^q)x, x > > g_A(|b|^2) - 2 < f_A(A_1^{\frac{q}{2}+1})x, x > > g_A(|b|^{\frac{q}{2}+1}) \right\} + \]

\[ + B_2 \left( \frac{1}{p} \right) \left\{ (2 - q)^2 < S_1(A_1^{2+q}) \log^2(A_1)x, x > > g(|b|^{p+2}) + \right. \]

\[ + (p - 2)^2 < f(A_1^{2+q})x, x > > S_2(|b|^{p+2}) \log^2(|b|) \right\} + \]

\[ + 2(2 - q)(p - 2) < S_3(A_1^{2+q}) \log(A_1)x, x > > S_4(|b|^{p+2}) \log(|b|) \leq \]
\[+(p-2)^2 < f(A_1^{2+q})x, x > S_2(|b|^{p+2}) \log^2(|b|) + \]
\[+2(2-q)(p-2) < S_3(A_1^{2+q}) \log(A_1)x, x > S_4(|b|^{p+2}) \log(|b|).\]

Applying now for \(|b|\) fixed the theorem 1, for any \(y \in \mathcal{K}\) we get the desired inequality. \(\square\)

In the last result we use the same inequality (1.1) but we will replace the variable \(a\) and \(b\) by other suitable variable and then we will follow the same steps like before.

**Theorem 4.** Let \(f(z) = \sum_{n=0}^{\infty} p_n z^n, \ g(z) = \sum_{n=0}^{\infty} q_n z^n\) be the power series with real coefficients and convergent on the open disk \(D(0, R), \ 0 < R < 1.\) If \(p, q\) are real numbers with \(p > 1, \frac{1}{p} + \frac{1}{q} = 1\) and \(A_1\) is a positive definite operator on the Hilbert space \(\mathcal{H}, \ A_1 \in B(\mathcal{H})\) with \(Sp(A_1) \subseteq (0, R)\) and \(B\) is a positive operator on the pseudo-Hilbert space \(\mathcal{K}, \ B \in B_\mathcal{K}(\mathcal{K})\) with \(Sp(B) \subseteq (0, R)\) then we have

\[< f_A(A_1^2) x, x > [g_A(B^2) y, y] + r \{< f_A(A_1^2) x, x > [g_A(B^p) y, y]\]
\[+ < f_A(A_1^2) x, x > [g_A(B^q) y, y] - 2 < f_A(A_1^2) x, x > [g_A(B^{p+q}) y, y] + A_2 \left(\frac{1}{p}\right) M \leq \]
\[\leq < f_A(A_1^2) x, x > [g_A(B^q) y, y]\]
\[\leq < f_A(A_1^2) x, x > [g_A(B^2) y, y] + (1 - r) < f_A(A_1^2) x, x > [g_A(B^p) y, y] + \]
\[+ < f_A(A_1^2) x, x > [g_A(B^q) y, y] - 2 < f_A(A_1^2) x, x > [g_A(B^{p+q}) y, y] + B_2 \left(\frac{1}{p}\right) M\]

where
\[M = (p-q)^2 < f(A_1^2)x, x > [S_1(B^{p+q}) \log^2(B) y, y]\]
and \(S_1(a) = a f'_A(a) + a^2 f''_A(a), \ S_2(a) = a g'_A(a) + a^2 g''_A(a), \ S_3(a) = a f'_A(a), \ S_4(a) = a g'_A(a)\) for each \(x \in H\) and \(y \in \mathcal{K}, \) where \(\lambda \in [0, 1], \ r = \min\{\lambda, 1 - \lambda\}.\)

**Proof.** This time we consider, in inequality (1.1) \(|a|^{2j} |b|^k\) instead of \(a\) and \(|a|^{2j} |b|^k\) instead of \(b\) where \(k, j \in \mathbb{N}\) and we obtain:

\[|a|^{2j} |b|^{2k} + r \left(|a|^{2j} |b|^k + |a|^{2j} |b|^k - 2 |a|^{2j} |b|^{k+2}\right) + \]
\[+ A_2 \left(\frac{1}{p}\right) |a|^{4j} |b|^{k(p+q)} \log^2 \left(\frac{|a|^{2j} |b|^k}{|a|^{2j} |b|^k}\right) \leq \frac{|a|^{2j} |b|^{k} p}{p} + \frac{|a|^{2j} |b|^q}{q} \leq \]
\[\leq |a|^{2j} |b|^{2k} + (1 - r) \left(|a|^{2j} |b|^k + |a|^{2j} |b|^k - 2 |a|^{2j} |b|^{k+2}\right) + \]
\[+ B_2 \left(\frac{1}{p}\right) |a|^{4j} |b|^{k(p+q)} \log^2 \left(\frac{|a|^{2j} |b|^k}{|a|^{2j} |b|^k}\right).\]

If we multiply the inequality with positive quantities \(|p_j||q_k|\) and sum over \(j\) and \(k\) from 0 to \(n,\) we find that

\[\sum_{k=0}^{n} \sum_{j=0}^{n} |p_j||q_k| |a|^{2j} |b|^{2k} + r \sum_{k=0}^{n} \sum_{j=0}^{n} |p_j||q_k| \left(|a|^{2j} |b|^k + |a|^{2j} |b|^k - 2 |a|^{2j} |b|^{k+2}\right) + \]
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\[ + A_2 \left( \frac{1}{p} \right) \sum_{k=0}^{n} \sum_{j=0}^{n} \left| p_j \right| \left| q_k \right| k^2 (p - q)^2 |a|^{4j} |b|^{k(p+q)} \log^2 (|b|) \leq \]

\[ \leq \sum_{k=0}^{n} \sum_{j=0}^{n} \left| p_j \right| \left| q_k \right| \frac{|a|^{2j} |b|^{kp}}{p} + \sum_{k=0}^{n} \sum_{j=0}^{n} \left| p_j \right| \left| q_k \right| \frac{|a|^{2j} |b|^{kq}}{q} \leq \]

\[ \leq \sum_{k=0}^{n} \sum_{j=0}^{n} \left| p_j \right| \left| q_k \right| |a|^{2j} |b|^{2k} + (1-r) \sum_{k=0}^{n} \sum_{j=0}^{n} \left| p_j \right| \left| q_k \right| \left( |a|^{2j} |b|^{kp} + |a|^{2j} |b|^{kq} - 2|a|^{2j} |b|^{k\frac{p+q}{2}} \right) + \]

\[ + B_2 \left( \frac{1}{p} \right) \sum_{k=0}^{n} \sum_{j=0}^{n} \left| p_j \right| \left| q_k \right| k^2 (p - q)^2 |a|^{4j} |b|^{k(p+q)} \log^2 (|b|) . \]

By the same reason as in previous theorem using the hypothesis we find the desired inequality.

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References


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