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p-Liar's Domination in a Graph

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Abstract

A set $S \subseteq V(G)$ is a *p-liar's dominating set* (*plds*) of graph G if (i) $|N_G(v) \cap S| \geq 2$ for every $v \in V(G) \setminus S$ (that is, S is a 2- dominating set of G), and (ii) $|[N_G(u) \cup N_G(v)] \cap S| \geq 3$ for any two distinct vertices $u, v \in V(G) \setminus S$. The *p-liar's domination number* of G, denoted by $\gamma_{pLR}(G)$, is the smallest cardinality of a p-liar's dominating set of G. In this paper, we study the concept of p-liar's domination of a graph G and investigate it for graphs resulting from the binary operations join and corona.

1 Introduction

Let $G = (V(G), E(G))$ be a simple connected graph. The neighborhood of $v \in V(G)$ is the set $N_G(v) = \{x \in V(G) : xv \in E(G)\}$. A vertex

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v in a graph G is *isolated* if $deg_G v = 0$; it is an *end-vertex* or *leaf* if $deg_G v = 1$. Denote by $\mathcal{I}(G)$ and $\mathcal{L}(G)$ the set of all isolated vertices and leaves of G, respectively. Let $S \subseteq V(G)$ of a graph G. A vertex w is an *external private neighbor* (abbreviated *epn*) of $v \in S$ if $w \in V(G) \setminus S$ and $N(w) \cap S = \{v\}$. The set of all external private neighbors of v is denoted by $epn(v; S)$.

A set $S \subseteq V(G)$ is a *dominating set* of G if for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $uv \in E(G)$, that is, $N_G[S] = V(G)$. The *domination number* of G, denoted by $\gamma(G)$, is the smallest cardinality of a dominating set of G. Any dominating set in G of cardinality $\gamma(G)$ is referred to as a γ-set of G. A subset S of $V(G)$ is called an *almost dominating set* of G if $|V(G) \setminus N[S]| \leq 1$. The *a-domination number* of G, denoted by $\gamma_a(G)$, is the smallest cardinality of an almost dominating set of G. An almost dominating set of G with cardinality $\gamma_a(G)$ is referred to as a γ_a-set of G. A set $S \subseteq V(G)$ is a *liar's dominating set* (*lds*) of G if (i) $|N_G[v] \cap S| \geq 2$ for every $v \in V(G)$ (that is, S is a double dominating set of G), and (ii) $|(N_G[u] \cup N_G[v]) \cap S| \geq 3$ for any two distinct vertices $u, v \in V(G)$. A set $S \subseteq V(G)$ is a *p-liar's dominating set* (*plds*) of G if (i) $|N_G(v) \cap S| \geq 2$ for every $v \in V(G) \setminus S$ (that is, S is a 2-dominating set of G), and (ii) $|[N_G(u) \cup N_G(v)] \cap S| \geq 3$ for any two distinct vertices $u, v \in V(G) \setminus S$. The *p-liar's domination number* of G, denoted by $\gamma_{pLR}(G)$, is the smallest cardinality of a p-liar's dominating set of G. Any subset of $V(G)$ with cardinality $\gamma_{pLR}(G)$ is called a γ_{pLR}-set of G.

Domination in graph as well some of its variations can be found in [1]. The concept of *liar's domination* of a graph G was introduced by Slater and Roden in [3]. About five years later, Sterling [4] studied the concept for grid graphs. A similar study of the concept is also considered in [2].

2 Results

It is easy to see that the vertex set of a graph is p-liar's dominating set. Hence the following observation is immediate.

Remark 2.1 *Let G be a graph of order n. Then* $1 \leq \gamma_{pLR}(G) \leq n$.

Theorem 2.2 *Let G be a graph. Then*

(i) $\gamma_{pLR}(G) = 1$ *if and only if* $G \cong K_1$; *and*

(ii) $\gamma_{pLR}(G) = 2$ *if and only if* $G \in \{K_2, \overline{K}_2, K_3, P_3\}$.

Proof:

(*i*) Suppose $\gamma_{pLR}(G) = 1$ and let $S = \{x\}$ be a plds of G. If there exists $y \in V(G) \setminus \{x\}$, then $xy \in E(G)$ and $|N_G(x) \cap S| = 1$. Hence, S is not plds of G, contrary to our assumption. Thus, $V(G) = \{x\}$, that is, $G \cong K_1$. The converse is clear.

(*ii*) Let $S = \{x, y\}$ be a γ_{pLR}-set of G. Then by Definition and Theorem 2.3, $|V(G)| = 2$ or $|V(G)| = 3$. If $|V(G)| = 2$, then $G \cong K_2$ or $G \cong \overline{K}_2$. If $|V(G)| = 3$, then $G \cong P_3$ or $G \cong K_3$. The converse is straightforward.

Remark 2.3 *Let G be a connected graph of order $n \geq 4$. Then $\gamma_{pLR}(G) \geq 3$.*

Remark 2.4 *Let G be a graph. Then $\mathcal{I}(G) \cup \mathcal{L}(G) \subseteq S$ for every liar's dominating set S of G.*

Theorem 2.5 *Let n be a positive integer. Then*

(*i*) $$\gamma_{pLR}(K_n) = \begin{cases} 1, & \text{if } n = 1 \\ 2, & \text{if } n = 2 \text{ or } n = 3 \\ 3, & \text{if } n \geq 4; \end{cases}$$

(*ii*) $\gamma_{pLR}(P_n) = \lceil \frac{n+1}{2} \rceil$ *for* $n \geq 1$*; and*

(*iii*) $$\gamma_{pLR}(C_n) = \begin{cases} 3, & \text{if } n = 4 \\ \frac{n+1}{2}, & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even } (n \neq 4). \end{cases}$$

Proof:

(*i*) Let $G = K_n$. If $n = 1$, then $\gamma_{pLR}(G) = 1$ by Theorem 2.2. If $n = 2$ or 3, then $\gamma_{pLR}(G) = 2$ by Theorem 2.3. Suppose $n \geq 4$. Then $\gamma_{pLR}(G) \geq 3$ by Remark 2.4. Pick any distinct vertices x, y, z of G and let $S = \{x, y, z\}$. Then S is a p-liar's dominating set of G. Thus, $\gamma_{pLR}(G) = 3$.

(*ii*) Clearly, $\gamma_{pLR}(P_1) = 1$, and $\gamma_{pLR}(P_2) = \gamma_{pLR}(P_3) = 2 = \lceil \frac{3}{2} \rceil = \lceil \frac{4}{2} \rceil$ by Theorem 2.2 and Theorem 2.3, respectively. Let $n \geq 4$ and let $P_n = [x_1, x_2, ..., x_{n-1}, x_n]$. Let S be a γ_{pLR}-set of G. Then $x_1, x_n \in S$ by Remark 2.4. Also, $x_2 \notin S$ or $x_{n-1} \notin S$. Assume that $x_2 \notin S$ and consider the following cases:

Case 1: $n = 2k$ for some positive integer $k \geq 2$

Then $S = \{x_1, x_3, ..., x_{2(k-1)+1}, x_n\}$. Hence,

$$\gamma_{pLR}(G) = |S| = k + 1 = \frac{n}{2} + 1 = \frac{n+2}{2}.$$

Case 2: $n = 2k + 1$ for some positive integer $k \geq 2$

Then $S = \{x_1, x_3, ..., x_{2(k-1)+1}, x_n\}$. Hence,

$$\gamma_{pLR}(G) = |S| = k + 1 = \frac{n-1}{2} + 1 = \frac{n+1}{2}.$$

Accordingly, $\gamma_{pLR}(G) = \left\lceil \frac{n+1}{2} \right\rceil$.

(iii) Clearly, $\gamma_{pLR}(C_3) = \frac{3+1}{2} = 2$ and $\gamma_{pLR}(C_4) = 3$ by Theorem 3.3. Suppose $n \geq 5$ and let $C_n = [x_1, x_2, ..., x_n, x_1]$. Let S be γ_{pLR}-set of C_n and assume that $x_1 \in S$. Then $x_n \in S$ or $x_{n-1} \in S$. Assume $x_n \in S$ and consider the following cases:

Case 1: $n = 2k$ for some positive integer $k \geq 2$

Then $S = \{x_1, x_3, ..., x_{2(k-1)-1}, x_{2(k-1)+1}\}$. Thus, $|S| = k = \frac{n}{2}$. Since $x_n \in S, |S| = k + 1 = \frac{n}{2} + 1 = \frac{n+2}{2}$. Since $\frac{n}{2} < \frac{n+2}{2}$, it follows that $\gamma_{pLR}(G) = \frac{n}{2}$.

Case 2: $n = 2k + 1$ for some positive integer $k \geq 2$

Then $S = \{x_1, x_3, ..., x_{2(k-1)}, x_n\}$. Hence, $\gamma_{pLR}(G) = |S| = k + 1 = \frac{n-1}{2} + 1 = \frac{n+1}{2}$.

This completes the proof of the theorem. ■

Theorem 2.6 *Let G be a graph of order $n \geq 3$. Then $\gamma_{pLR}(G) = n$ if and only if for each component $\mathcal{C}$ of G, $\mathcal{C} \cong K_1$ or $\mathcal{C} \cong K_2$.*

Proof: Suppose $\gamma_{pLR}(G) = n$. Let $\mathcal{C}$ be a component of G. Suppose that $\mathcal{C}$ is neither K_1 nor K_2. Then $|V(\mathcal{C})| \geq 3$ and there exists $v \in V(\mathcal{C})$ such that $deg_G v \geq 2$. Let $S = V(G) \setminus \{v\}$. Then S is p-liar's dominating set of G. Thus $\gamma_{pLR}(G) \leq |S| = n - 1$, contrary to our assumption. Therefore, $\mathcal{C} \cong K_1$ or $\mathcal{C} \cong K_2$.

The converse follows from Remark 2.4. ■

Theorem 2.7 *Let G be a graph of order $n \geq 4$. Then $\gamma_{pLR}(G) = 3$ if and only if there exists $S \subseteq V(G)$ with $|S| = 3$ such that S is a 2-dominating set and for each pair of distinct vertices $u, v \in V(G) \setminus S$, either $N_G(u) \cap S \neq N_G(v) \cap S$ or $N_G(u) \cap S = N_G(v) \cap S = S$.*

Proof: Suppose $\gamma_{pLR}(G) = 3$. Let S be γ_{pLR}-set of G. Then $|S| = 3$ and by Definition of p-liar's (i), S is a 2-dominating set of G. Let $u, v \in V(G) \setminus S$ with $u \neq v$. Suppose $N_G(u) \cap S = N_G(v) \cap S$. Since S is p-liar's dominating set, $|(N_G(u) \cap S) \cup (N_G(v) \cap S)| = |(N_G(u) \cup N_G(v)) \cap S| = |N_G(u) \cap S| = 3$ by Definition p-liar's (ii). This implies that $N_G(u) \cap S = N_G(v) \cap S = S$.

For the converse, suppose there exists $S \subseteq V(G)$ satisfying the given conditions. Then S is a p-liar's dominating set of G. Since $n \geq 4$, S is a γ_{pLR}-set of (G) by Remark 2.3. Thus, $\gamma_{pLR}(G) = |S| = 3$. ■

Corollary 2.8 *Let G be a connected graph of order $n = 4$. Then $\gamma_{pLR}(G) = 3$.*

3 p-Liar's Domination in the Join of Graphs

The *join* of two graphs G and H is the graph $G+H$ with vertex-set $V(G+H) = V(G) \cup V(H)$ and edge-set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

Theorem 3.1 *Let G be a graph of order $n \geq 2$ and $K_1 = \langle\{v\}\rangle$. Then $S \subseteq V(K_1 + G)$ is a p-liar's dominating set of $K_1 + G$ if and only if one of the following holds:*

(*i*) *S is a p-liar's dominating set of G, where $S = V(G)$ whenever $n = 2$ or 3.*

(*ii*) *$S = S_1 \cup \{v\}$, where S_1 is a dominating set of G and $|epn(x; S_1)| \leq 1$ for all $x \in S_1$.*

Proof: Suppose S is plds of $K_1 + G$. Consider the following cases:

Case 1. $v \notin S$

Then $S \subseteq V(G)$. Since S is a p-liars dominating set of $K_1 + G$, S is a plds of G. If $n = 2$, then $K_1 + G$ is P_3 or K_3. It follows from Theorem 2.3 that $S = V(G)$. If $n = 3$, then $\gamma_{pLR}(K_1 + G) = 3$ by Corollary 2.8. Hence, $S = V(G)$.

Case 2. $v \in S$

Then $S = S_1 \cup \{v\}$, where $S_1 \subseteq V(G)$. Let $z \in V(G) \setminus S_1$. Since S is a 2-dominating set of $K_1 + G$ and $v \in S \cap N_{K_1+G}(z)$, it follows that $zy \in E(G)$ for some $y \in S_1$. Hence, S_1 is a dominating set of G. Now, let $x \in S_1$ and suppose that $|epn(x; S_1)| \geq 2$. Let $a, b \in epn(x; S_1)$, where $a \neq b$. Then $N_G(a) \cap S_1 = N_G(b) \cap S_1 = \{x\}$. Thus, $N_{K_1+G}(a) \cap S = N_{K_1+G}(b) \cap S = \{x, v\}$. This implies that S does not satisfy (*ii*) of Definition of p-liar's, contradicting our assumption that S is a plds of $K_1 + G$. Therefore, $|epn(x; S_1)| \leq 1$.

For the converse, suppose that (*i*) holds. Then, clearly, S is a *plds* of $K_1 + G$. Next, suppose that (*ii*) holds. Let $u \in V(K_1 + G) \setminus S$. Then $u \in V(G) \setminus S_1$. Since S_1 is a dominating set of G, $|N_G(u) \cap S_1| \geq 1$. Hence, $|N_{K_1+G}(u) \cap S| = |(N_G(u) \cap S_1) \cup \{v\}| \geq 2$. This shows that S is a 2-dominating set of K_1+G. Now, let $z, w \in V(K_1+G) \setminus S$. Then $z, w \in V(G) \setminus S_1$. If none of z and w is an external private neighbor of any element of S_1, then $|N_G(z) \cap S_1| \geq 2$ and $|N_G(w) \cap S_1| \geq 2$. Hence, $|(N_{K_1+G}(z) \cap S) \cup (N_{K_1+G}(w) \cap S)| \geq 3$. Suppose now that one of z and w, say z, is an external private neighbor of $x \in S_1$. Then, by assumption, $w \notin epn(x; S)$. This implies that $N_G(z) \cap S_1 = \{x\} \neq N_G(w) \cap S_1$.

Hence, $N_{K_1+G}(z) \cap S = \{x, v\} \neq N_{K_1+G}(w) \cap S$, showing that $|(N_{K_1+G}(z) \cup N_{K_1+G}(w)) \cap S| \geq 3$. Therefore, S is a plds of $K_1 + G$. ■

As a consequence of Theorem 3.1, we have the Corollary 3.2.

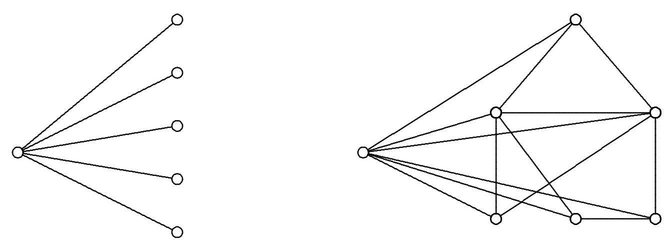
Corollary 3.2 *Let G be a graph of order $n \geq 4$. Then*

$$\gamma_{pLR}(K_1 + G) = min\{\gamma_{pLR}(G), \gamma^*(G) + 1\},$$

where $\gamma^*(G) = min\ \{|D| : D$ is a dominating set of G with $|epn(x; D)| \leq 1$ for all $x \in D\}$.

Proof: Let S be a γ_{PLR}-set and D be a γ^*-set of G. Then S and $S^* = D \cup \{v\}$ are plds of $K_1 + G$, by Theorem 3.1. Thus, $\gamma_{pLR}(K_1 + G) \leq min\{|S|, |S^*|\} = min\{\gamma_{pLR}(G), \gamma^*(G) + 1\}$. Next, let S be a γ_{pLR}-set of $K_1 + G$. By Theorem 3.1, S is a plds of G or $S = S_1 \cup \{v\}$, where S_1 is a dominating set of G with $|epn(x; S_1)| \leq 1$ for all $x \in S_1$. Hence, $\gamma_{pLR}(K_1 + G) = |S| \geq min\{\gamma_{pLR}(G), \gamma^*(G) + 1\}$. This proves the desired equality. ■

Example 3.3 *Consider the graphs $K_1 + \overline{K}_5$ and $K_1 + G$ in Figure 1.*



$K_1 + \overline{K}_5$ $K_1 + G$

Figure 1: The graphs $K_1 + \overline{K}_5$ and $K_1 + G$

Now, $\gamma_{pLR}(\overline{K}_5) = 5 = \gamma^*(\overline{K}_5)$. Then $\gamma_{pLR}(\overline{K}_5) = 5 < 1 + 5 = 1 + \gamma^*(\overline{K}_5)$. Hence, $\gamma_{pLR}(K_1 + G) = 5 = \gamma_{pLR}(\overline{K}_n)$. Also, $\gamma_{pLR}(G) = 4$ and $\gamma^*(G) = 2$. Thus $1 + \gamma^*(G) = 1 + 2 = 3 < 4 = \gamma_{pLR}(G)$. Hence, $\gamma_{pLR}(K_1 + G) = 3 = 1 + \gamma^*(G)$.

Theorem 3.4 *Let G and H be non-trivial connected graphs. Then $S \subseteq V(G + H)$ is a p-liar's dominating set of $G + H$ if and only if one of the following holds:*

(*i*) *S is a p-liar's dominating set of G.*

(*ii*) *S is a p-liar's dominating set of H.*

(*iii*) $|S \cap V(G)| \geq 3$ *and* $|S \cap V(H)| \geq 3$.

(*iv*) $S = S_1 \cup S_2$, *where* $S_1 \subseteq V(G)$ *and* $S_2 \subseteq V(H)$ *and satisfy the following:*

(*a*) S_1 *is a dominating set of* G *such that* $|S_1| \geq 2$ *and* $|epn(x; S_1)| \leq 1$ *for all* $x \in S_1$, *and*

(*b*) $|S_2| = 1$, *where* $|V(H) \setminus N_H[S_2]| \leq 1$ *whenever* $|S_1| = 2$.

(*v*) $S = S_1 \cup S_2$, *where* $S_1 \subseteq V(H)$ *and* $S_2 \subseteq V(G)$ *and satisfy the following:*

(*a*) S_1 *is a dominating set of* H *such that* $|S_1| \geq 2$ *and* $|epn(x; S_1)| \leq 1$, *for all* $x \in S_1$, *and*

(*b*) $|S_2| = 1$, *where* $|V(G) \setminus N_G[S_2]| \leq 1$ *whenever* $|S_1| = 2$.

(*vi*) $S = S_1 \cup S_2$, *where* $S_1 \subseteq V(G)$ *and* $S_2 \subseteq V(H)$ *and satisfy the following:*

(*a*) $|S_1| = 2$ *and* $|V(G) \setminus N_G[S_1]| \leq 1$, *and*

(*b*) $|S_2| = 2$ *and* $|V(H) \setminus N_H[S_2]| \leq 1$.

(*vii*) $S = S_1 \cup S_2$, *where* $S_1 \subseteq V(G)$ *and* $|S_1| = 2$ *and* $S_2 \subseteq V(H)$, $|S_2| \geq 3$ *and* $|V(H) \setminus N_H[S_2]| \leq 1$.

(*viii*) $S = S_1 \cup S_2$, *where* $S_1 \subseteq V(H)$ *and* $|S_1| = 2$ *and* $S_2 \subseteq V(G)$, $|S_2| \geq 3$ *and* $|V(G) \setminus N_G[S_2]| \leq 1$.

Proof: Suppose S is a plds of $G + H$. If $S \cap V(H) = \varnothing$ or $S \cap V(G) = \varnothing$, then S is a plds of G or H. Thus, (*i*) or (*ii*) holds. Now, suppose that $S_1 = S \cap V(G) \neq \varnothing$ and $S_2 = S \cap V(H) \neq \varnothing$. If $|S_1| \geq 3$ and $|S_2| \geq 3$, then (*iii*) holds. Consider the following cases:

Case 1. $|S_1| \geq 2$ and $|S_2| = 1$ or $|S_1| = 1$ and $|S_2| \geq 2$

Suppose that $|S_1| \geq 2$ and $|S_2| = 1$. Let $x \in V(G) \setminus S_1$. Since S is a 2-dominating set of $G + H$, $|N_{G+H}(x) \cap S| = |S_2| + |N_G(x) \cap S_1| = 1 + |N_G(x) \cap S_1| \geq 2$. This implies that $|N_G(x) \cap S_1| \geq 1$. Hence, S_1 is a dominating set of G. Suppose that $|epn(x; S_1)| \geq 2$. Let y, z be distinct external private neighbors of x in S_1. Then $N_{G+H}(y) \cap S = N_{G+H}(z) \cap S = \{x\} \cap S_2$. This implies that S does not satisfy Definition of p-liar's (*ii*), contrary to our assumption that S is p-liar's dominating set of $G + H$. Therefore, $|epn(x; S_1)| \leq 1$ for all $x \in S_1$. Next, suppose that $|S_1| = 2$. Suppose further that $|V(H) \setminus N_H[S_2]| \geq 2$. Then there exist distinct vertices $a, b \in V(H) \setminus N_H[S_2]$ such that $[N_{G+H}(a) \cap N_{G+H}(b)] \cap S| = |S_1| = 2$, contrary to our assumption that S is a plds of $G + H$. Therefore, (*iv*) holds. Similarly, (*v*) holds if $|S_1| = 1$ and $|S_2| \geq 2$.

Case 2. $|S_1| = 2$ and $|S_2| = 2$

Suppose that $|V(G) \setminus N[S_1]| \geq 2$. Then there exist $x, y \in V(G)$ such that $x, y \notin N[S_1]$. This implies that $[N_{G+H}(x) \cap N_{G+H}(y)] \cap S| = 2$. Thus, S does not satisfy the Definition of p-liar's (ii), contrary to our assumption that S is p-liar's dominating set of $G + H$. Hence, $|V(G) \setminus N[S_1]| \leq 1$. Similarly, $|V(H) \setminus N[S_2]| \leq 1$.

Case 3. $|S_1| = 2$ and $|S_2| \geq 3$ or $|S_1| \geq 3$ and $|S_2| = 2$

Suppose that $|S_1| = 2$ and $|S_2| \geq 3$ and suppose further that $|V(H) \setminus N[S_2]| \geq 2$. Then, $[N_{G+H}(x) \cap N_{G+H}(y)] \cap S| = |S_1| = 2$ for some $x, y \in V(H) \setminus N[S_2]$, contrary to the fact that S is a p-liar's dominating set of $G+H$. Thus, $|V(H) \setminus N[S_2]| \leq 1$, showing that (vi) holds. Similarly, (vii) holds if $|S_1| \geq 3$ and $|S_2| = 2$.

The converse is clear. ■

As a consequence of Theorem 3.4, we have the next results.

Remark 3.5 *Let G and H be non-trivial connected graphs. Then* $3 \leq \gamma_{pLR}(G + H) \leq 6$.

Corollary 3.6 *Let G and H be non-trivial connected graphs. Then $\gamma_{pLR}(G + H) = 3$ if and only if one of the following holds:*

(i) $\gamma_{pLR}(G) = 3$;

(ii) $\gamma_{pLR}(H) = 3$;

(iii) $\gamma_a(H) = 1$ *and* $\gamma^*(G) \leq 2$, *where* $\gamma^*(G) = min\ \{|S_2| : S_2$ *is a dominating set of G with* $|epn(x; S_2)| \leq 1$ *for all* $x \in S_2\}$; *or*

(iv) $\gamma_a(G) = 1$ *and* $\gamma^*(H) \leq 2$, *where* $\gamma^*(H) = min\ \{|S_1| : S_1$ *is a dominating set of H with* $|epn(x; S_1)| \leq 1$ *for all* $x \in S_1\}$.

Corollary 3.7 *Let G and H be non-trivial connected graphs such that $\gamma_{pLR}(G + H) \neq 3$. Then $\gamma_{pLR}(G + H) = 4$ if and only if one of the following holds:*

(i) $\gamma_{pLR}(G) = 4$;

(ii) $\gamma_{pLR}(H) = 4$;

(iii) $\gamma_a(G) \leq 2$ *and* $\gamma_a(H) \leq 2$;

(iv) $\gamma^*(H) = 3$, *where* $\gamma^*(H) = min\ \{|S_2| : S_2$ *is a dominating set of H with* $|epn(x; S_2)| \leq 1$ *for all* $x \in S_2\}$; *or*

(*v*) $\gamma^*(G) = 3$, *where* $\gamma^*(G) = min\ \{|S_2| : S_2$ *is a dominating set of* G *with* $|epn(x; S_2)| \leq 1$ *for all* $x \in S_2\}$.

Corollary 3.8 *Let* G *and* H *be non-trivial connected graphs such that* $\gamma_{pLR}(G + H) > 4$. *Then* $\gamma_{pLR}(G + H) = 5$ *if and only if one of the following holds:*

(*i*) $\gamma_{pLR}(G) = 5$;

(*ii*) $\gamma_{pLR}(H) = 5$;

(*iii*) $\gamma_a(H) = 3$;

(*iv*) $\gamma_a(G) = 3$;

(*v*) $\gamma^*(G) = 4$, *where* $\gamma^*(G) = min\ \{|S_2| : S_2$ *is a dominating set of* G *with* $|epn(x; S_2)| \leq 1$ *for all* $x \in S_2\}$; *or*

(*vi*) $\gamma^*(H) = 4$, *where* $\gamma^*(H) = min\ \{|S_2| : S_2$ *is a dominating set of* H *with* $|epn(x; S_2)| \leq 1$ *for all* $x \in S_2\}$.

4 p-Liar's Domination in the Corona of Graphs

For every $v \in V(G)$, denote by H^v the copy of H whose vertices are attached one by one to the vertex v. Denote by $v + H^v$ the subgraph of the corona $G \circ H$ corresponding to the join $\langle \{v\} \rangle + H^v$.

Theorem 4.1 *Let* G *be non-trivial connected graph and* H *be any graph. Then* $C \subseteq V(G \circ H)$ *is a p-liar's dominating set of* $G \circ H$ *if and only if* $C = A \cup (\cup_{v \in A} S_v) \cup (\cup_{u \notin A} D_u)$, *where* $A \subseteq V(G)$, S_v *is a dominating set of* H^v *with* $|epn(x; S_v)| \leq 1$ *for each* $x \in S_v$ *and for each* $v \in A$, *and* D_u *is a p-liar's dominating set of* H^v *for each* $u \notin A$, *where* $D_u = V(H^v)$ *whenever* $|V(H)| = 2$ *or* 3.

Proof: Suppose C is a plds of G. Let $A = C \cap V(G)$. Pick any $v \in A$ and let $S_v = C \cap V(H^v)$. By Theorem 3.2.1, S_v is a dominating set of H^v and $|epn(x; S_v)| \leq 1$ for all $x \in S_v$. Next, let $u \notin A$ and set $D_u = C \cap V(H^v)$. By Theorem 3.2.1, D_u is a p-liar's dominating set of H^u, where $D_u = V(H^v)$ whenever $|V(H)| = 2$ or 3.

For the converse, suppose that C has the given form and the corresponding properties. Let $z \in V(G \circ H) \setminus C$ and let $w \in V(G)$ such that $z \in V(w + H^w)$. Suppose $z = w$. Then $z \notin A$. By assumption, D_z is a p-liar's dominating set of H^z, where $D_z = V(H^z)$ whenever $|V(H)| = 2$ or 3. Hence, $|N_{G \circ H}(z) \cap C| \geq |D_z| \geq 2$. Suppose $z \neq w$. Then $z \in V(H^w) \setminus C$. If $w \notin A$, then

$D_w = C \cap V(H^w)$ is a p-liar's dominating set of H^w. Since $z \notin D_w$, it follows that $|N_{G\circ H}(z) \cap C| = |N_{H^w}(z) \cap D_w| \geq 2$. If $w \in A$, then $S_w = C \cap V(H^w)$ is a dominating set of H^w by assumption. Thus, $|N_{G\circ H}(z) \cap C| \geq 1 + |N_{H^w}(z) \cap S_w| \geq 2$. Hence, C is a 2-dominating set of $G \circ H$.

Finally, let $a, b \in V(G \circ H) \setminus C$ $(a \neq b)$ and let $u, v \in V(G)$ such that $a \in V(u + H^u)$ and $b \in V(v + H^v)$. Consider the following cases:

Case 1. $u = v$

Suppose $u \in A$. Then, by assumption, S_u is a dominating set of H^u with $|epn(x; S_u) \leq 1$ for all $x \in S_u$. Thus, since $a, b \in V(H^u) \setminus S_u$,

$$|\,[N_{G\circ H}(a) \cup N_{G\circ H}(b)] \cap C| = |\,[N_{H^u}(a) \cup N_{H^u}(b)] \cap S_u| + 1 \geq 3.$$

Suppose now that $u \notin A$. Then $D_u = V(H^u) \cap C$ is a p-liar's dominating set of H^u, where $D_u = V(H^u)$ whenever $|V(H)| = 2$ or 3. Suppose that one of a and b, say a, is u. Then $b \in V(H^u) \setminus D_u$. Hence, by assumption, $|V(H)| \geq 4$. This implies that $|D_u| \geq 3$ and $|[N_{G\circ H}(a) \cup N_{G\circ H}(b)] \cap D_u| \geq 3$. So suppose that $a, b \in V(H^u) \setminus D_u$. Since D_u is a plds of H^u, it follows that

$$|[N_{G\circ H}(a) \cup N_{G\circ H}(b)] \cap C| = |[N_{H^u}(a) \cap N_{H^u}(b)] \cap D_u| \geq 3.$$

Case 2. $u \neq v$

Consider the following sub-cases:

Sub-case 1. $u, v \in A$

Then S_u and S_v are dominating sets of H^u and H^v, respectively, and $a \in V(H^u) \setminus S_u$ and $b \in V(H^v) \setminus S_v$. Thus,

$$|[N_{G\circ H}(a) \cup N_{G\circ H}(b)] \cap C| = 2 + |N_{H^u}(a) \cap S_u| + |N_{H^v}(b)] \cap S_v| \geq 4.$$

Sub-case 2. $u, v \notin A$

Then D_u and D_v are plds of H^u and H^v, respectively. Thus, $|[N_{G\circ H}(a) \cup N_{G\circ H}(b)] \cap C| \geq 4$.

Sub-case 3. $u \in A$ and $v \notin A$ or $u \notin A$ and $v \in A$.

Then, by assumptions, $|[N_{G\circ H}(a) \cup N_{G\circ H}(b)] \cap C| \geq 4$.

Accordingly, C is a plds of $G \circ H$. ■

The next result is a direct consequence of Theorem 4.1.

Corollary 4.2 *Let G be non-trivial connected graph of order $n \geq 4$ and H be any graph of order m. Then*

$$\gamma_{pLR}(G \circ H) = min\{|A| + |A|\gamma^*(H) + (n - |A|)\gamma_{pLR}(H) : A \subseteq V(G)\},$$

where $\gamma^*(H) = min\ \{|S_v| : S_v$ is a dominating set of H^v with $|epn(x; S_v)| \leq 1$ for each $x \in S_v$ and for each $v \in A\}$. In particular, $\gamma_{pLR}(G \circ H) \leq n\gamma_{pLR}(H)$.

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