Partition Games as Contests:
Numerical Simulation Results

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Abstract

The paper examines of the disjoint subsets of the strategies for the partition games to elucidate their “relative strength”, i.e. to define which strategies to gain the wins account in the games such as, for instance, Lotto game. It can be inferred that analysis of the disjoint subsets of the set $(n, m)$-partitions enable one to choice the strategies with high “winning ability”. Our focus is on the computing simulation of the elimination tournaments to check these assumptions. Actors in such elimination tournaments are the partitions and the rule are identical to that for a Lotto game. Computing simulation’s results demonstrated
that contestant using the strategies from specially designed disjoint sub-
sets of given \((n,m)\)-partitions would wins (with frequency more than
0.9) in the tournaments with all given partitions.

**Keywords:** game theory, colonel Blotto game, general Lotto game, coalition, tournament

## Preliminary Notes

In a wide range of disciplines, including computer science, political science, economics, management science, and the military sciences, there are environments that may be characterized as games of multiple contests with linkages [12]. Examples include innovation races that involve obtaining multiple, interrelated patents, counter-terrorism and information systems security efforts that involve allocating resources to the defense of networks of targets, multi-battle military conflicts involving allocating forces across space or time, and models of redistributive politics in which a budget is allocated across different constituencies in order to secure their political support. In each of these examples, the total payoff across the entire set of contests depends on how the individual contests enter into the players’ objectives and how the underlying technology links expenditures across the contests. These games have long attracted the attention of theorists and, more recently, have been the subject of experimental researches. For different kind of contests, especially in repeated contests where players compete in a sequence of battles and final victory is awarded as a function of the numbers of battle victories that the players accumulated, important features are structure of the players payoff functions, number of players, type of the players objective function and the contest rules. The examples of such battles can be found in many contexts, for instance, in sports, politics, warfare, R&D competition. So there are the vast of researches in this area (see for references [5]).

Special type of the contests are elimination tournaments models first such called the evolution games (see, for instance, [17]). For these games to win or lose is matter of a life or death literally. Detail discussing of the all contests type and almost comprehensive list of references could be find in [11]. In accordance of this work the focus of the most researches are on the role of various design aspects in contests, such as prize structure, sequencing, nesting, repetition, and many others.

Our main interest in the research of different types of interaction in which player expend effort in trying to get ahead of their rival. This problem was discussed earlier, see [11] and [15] for further references. Model of different types of interaction suppose mainly different effort level which choice each player. We will try to investigate the set of all pure strategies in a game (feasible vec-
tor type) to find the subset of strategies with maximal “winning ability”. Such approach seems extremely vague to obtain some general results in the game theory. Therefore, our field of interest is the symmetric antagonistic game with constant sum such as games of Colonel Blotto or Colonel Lotto.

1 Some Related Works and Main Purpose of the Work

Resource allocation plays a central role in both economics and politics. A well-studied game-theoretic representation of the resource allocation problem is the Blotto game [3], which models the problem as a two-player antagonistic game. In the canonical “Colonel Blotto game”, there are \( m \) “battlefields”. The two players have fixed endowments of resources, e.g., troops or money, which they must simultaneously allocate to each of the \( 2m \) battlefields. The winner of each battlefield is determined according to which player allocated the greater amount of resources to that battlefield. The standard objective function for the players is to win \( m/2 + 1 \) or more (a majority) of the battlefields. An alternative objective could be to win a many battlefields as possible. The Nash equilibrium of this deterministic version of the game are numerous and are in mixed strategies.

Colonel Blotto is a zero sum game, when one player wins, the other player loses. Games has not a straightforward solution, but has multiple mixed strategy equilibrium. For example, in one of the first work [7] was found that Colonel Blotto game with \( m \) fronts and total resource \( n \) has a mixed strategy equilibrium in which the marginal distributions are uniform on \( [0, 2n/m] \) all fronts. The best known results has been arrived in work [16], which has successfully characterized the unique equilibrium payoffs for all configurations of resource asymmetry, and the equilibrium resource allocation strategies (for most configurations) of a constant-sum Colonel Blotto game with \( m \geq 3 \) battlefields. It worth notes here that all pure strategies for Colonel Blotto could be present as partition number \( n \) on \( m \) parts (for more details explanation see section 1).

One of version of Colonel Blotto game became so called Lotto game and in particularly, Colonel Lotto and General Lotto games [8], and Captain Lotto game [9]. The only difference between Colonel Lotto game and Colonel Blotto game that battlefields for Colonel Lotto game are assumed to be indistinguishable. Therefore, it is necessary to compare of the player’s results for all permutations of the battlefields. In the case of General Lotto games, except assumption about indistinguishable of the battlefields, the players decide which fraction of the battlefields different amounts of the resources will be assigned to each. Thus, budget constraints of the players are expressed in terms
of expected values rather than in terms of total amounts of the resources. Therefore, the strategies of the players are the probability distributions over possible amounts of the resources. If these amounts are allowed to take any (non-negative) real value, then these probability distributions are over non-negative real numbers and the game is called continuous. If these amounts take integer values, then the game is discrete.

In mathematical theory game, the emphasis is on equilibrium problem: finding pure strategies of equilibrium or Nash equilibrium. Most of these problems have been decided for Colonel Blotto game and for different versions of Lotto game. However, these results through generality more often than not can lose their practical meaning. It is difficult to find the real games in which contestants could be used the mixed strategies to get game value, even if for no other reason than time constraints. Seems more useful to find the rank of the all pure strategies to order of their winning ability. However, it is necessary to find the ranks by some tractable algorithm because the power of the strategies set grows exponentially with n and m. For example the set of all \((100,10)\)-partitions contains 6292069 partitions. The main aim of our approach is to find such the subset of the all pure strategies for arbitrary discrete Lotto game that “better” (with point of view winning ability) of any other subset of the strategies by some tractable algorithm. (There “winning ability” is defined as the part of the wins during some predefine tournaments).

Actually algorithmic approaches in the game theory has much to contribute to the basic debate about solutions concept of the game [13] and [6]. Indeed if some concept is not efficiently computable seems that its credibility as a prediction of the game process could be lost.

The computing experiments are presented in the work has a one purpose: simulation of the non-symmetric (with point of the number of the pure strategies set) two-player Lotto game with point of quantitative estimate for “winning ability” of the all pure strategies. The program toolbox [2] is specially designed to conduct the experiments to partition games.

The remainder of this paper is organized as follows.

In the section 2 “Partitions Games”, the analysis tools for partitions “strength” ordering are introduced. Besides two important characteristic of \((n,m)\)-partitions — balance and peculiar resource — are presented. Both of them weigh with an ability of the partition to win in contests. The distribution of these characteristics values for all \((100,10)\)-partitions are shown.

In section 3 “Approximation Results for the Lotto Game”, we formulated of the basic proposition to numerical simulation of the partitions game — the tractable algorithm to evaluate of the pay function of Lotto game.

The section 4 “Tournaments: Experimental Results” is a basic. The numerical and graphical results of the computing simulation testifies that in the game tournaments there are the subsets of the partitions with obviously win-
ning ability.

In “Discussions” applications of our approach is discussed.

2 Partitions Games

Application of the language of combinatorial algorithms (see [10]) is best way for our goals. There all type of partitions be described in terms of the ways that a given number of balls — are labeled or unlabeled — can be placed into a given numbers of the urns (also labeled or unlabeled). A partition of \( n \) be defined as a sequence of non-negative integers \( a_1 \geq a_2 \geq \cdots \) such that \( n = a_1 + a_2 + \cdots \). The number of nonzero terms \( m \) is called the numbers of parts.

Hereafter we will use \((n,m)\)-partition as denotation any partitions of number \( n \) on the \( m \) urn exactly and two case: balls are unlabeled but urns are labeled (for model of Blotto game) and neither balls nor urns are labeled (for model of Lotto game).

For example, all partitions of 8 balls into 3 urns (in reverse lexicographic order started with \( n \)) are

\[
(2.1) \quad 8, 71, 62, 611, 53, 521, \ldots, 2111111, 11111111.
\]

There are a lot of asymptotic approximation for \( p(n) \) — number of all partition for \( n \). One of a relative rough approximation is

\[
(2.2) \quad p(n) = \frac{e^{\pi \sqrt{2n/3}}}{4n\sqrt{3}} \left(1 + \mathcal{O}(n^{-\frac{3}{2}})\right).
\]

For example, \( p(100) \) has the exact value 190 569 292 [10], (2.2) tells that \( p(n) \approx 1.993 \cdot 10^8 \). Most of all numerical experiments in our research is completed for \((100, 10)\)-partitions. Naturally, number of \((n,m)\)-partitions depends from \( m \) and for \((100, 10)\)-partition maximum is about 90 million for \( m = 18 \). The exact value for \((100, 10)\)-partitions is 6 292 069 [2].

The formal version of the Colonel Blotto game and Colonel Lotto game can be written as two-player, zero-sum game. Let \((n,m)\)-partition \( \alpha = \{a_1, a_2, \ldots, a_m\} \) be a pure strategy of player \( A \) and \((n,m)\)-partition \( \beta = \{b_1, b_2, \ldots, b_m\} \) — pure strategy of player \( B \). So both players have the same available budget.

The payoff to \( \alpha \) against \( \beta \) for Colonel Blotto game is

\[
(2.3) \quad B(\alpha, \beta) = \sum_{i=1}^{m} \text{sign}(x_i - y_i),
\]
Because for Colonel Lotto game the urns are unlabeled, the payoff to $\alpha$ against $\beta$ is

\[(2.4) \quad L(\alpha, \beta) = \sum_{\alpha \in \Theta_\alpha} \sum_{i=1}^{m} \text{sign}(x_i - y_i).\]

where $\Theta_\alpha$ is the set of the all permutations for $\alpha$.

So in general case for getting of $L(\alpha, \beta)$ it is necessary to compute $m!$ values of $B(\alpha, \beta)$.

For example, let two partition from $(8, 3)$-partitions are $\alpha = 4, 2, 2$ and $\beta = 5, 3, 0$. In order to compute (2.4) it is necessary to compute $3! = 6$ values of the functions (2.3), i. e. for all permutations urns of $\alpha$ (tb. 1).

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>530</th>
<th>503</th>
<th>035</th>
<th>053</th>
<th>305</th>
<th>350</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>422</td>
<td>422</td>
<td>422</td>
<td>422</td>
<td>422</td>
<td>422</td>
</tr>
<tr>
<td>$B(\alpha, \beta)$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$L(\alpha, \beta)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

Thus the payoff $L(\alpha, \beta)$ depends not only from $\alpha$ and $\beta$, but from probability distribution of the permutations set.

Our main aim is to estimate payoff $L(\alpha, \beta)$ without computing of the all $m!$ values $B(\alpha, \beta)$.

We define two types of interaction matrices $I^*$ ($m \times 4$) and $I$ ($m \times m$).

**Definition 2.1** Potential resource $S^*(\alpha, \beta)$ is

\[(2.5) \quad S^*(\alpha, \beta) = \sum_{i=1}^{m} \sum_{j=1}^{m} \text{sign}^*(a_i - b_j),\]

where $\text{sign}^*(a_i - b_j) = 1$ if $a_i > b_j$ else 0.

Accordingly, potential resource $S^*(\beta, \alpha)$ is

\[S^*(\beta, \alpha) = \sum_{j=1}^{m} \sum_{i=1}^{m} \text{sign}^*(b_j - a_i).\]

The value of both potential resources $S^*(\alpha, \beta)$ and $S^*(\beta, \alpha)$ can be represented by matrix $I^*$ ($m \times 4$). For example, let $n = 63$, $m = 7$, $\alpha = \{1, 3, 5, 7, 9, 11, 27\}$ and $\beta = \{2, 4, 6, 8, 10, 12, 21\}$. Matrix $I^*$ ($7 \times 4$) for pure strategies $\alpha$ and $\beta$ is presented in tb. 2.

In this case
Table 2: Matrix $I^*$ \((7 \times 4)\) for $\alpha = \{1, 3, 5, 7, 9, 11, 27\}$ and $\beta = \{2, 4, 6, 8, 10, 12, 21\}$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>21</th>
<th>12</th>
<th>10</th>
<th>8</th>
<th>6</th>
<th>4</th>
<th>2</th>
<th>$S^*(\alpha, \beta) = 27$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R^*(\alpha, \beta)$</td>
<td>6</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$\beta$</td>
<td>27</td>
<td>11</td>
<td>9</td>
<td>7</td>
<td>5</td>
<td>3</td>
<td>1</td>
<td>$S^*(\beta, \alpha) = 22$</td>
</tr>
<tr>
<td>$R^*(\beta, \alpha)$</td>
<td>7</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

\[(2.6) \quad R^*(\alpha, \beta) = \{ \sum_i \text{sign}(a_1 - b_i), \ldots, \sum_i \text{sign}(a_m - b_i) \}, \]

\[R^*(\beta, \alpha) = \{ \sum_i \text{sign}(b_1 - a_i), \ldots, \sum_i \text{sign}(b_m - a_i) \} \]

Notice that for tb. 2 $S^*(\alpha, \beta) + S^*(\beta, \alpha) = m^2$ but in general

\[(2.7) \quad S^*(\alpha, \beta) + S^*(\beta, \alpha) = m^2 - e, \]

where $e = |\alpha \cap \beta|$, i.e. a number of the coincident values in the partitions $\alpha$ and $\beta$.

Square matrix $I$ \((m \times m)\) for partitions $\alpha$ and $\beta$, where $c_{ij} = \text{sign}(a_i - b_j)$ for each matrix entries $c_{ij} \in I$.

**Definition 2.2** Relative strength $S(\alpha, \beta)$ is

\[(2.8) \quad S(\alpha, \beta) = \sum_i \sum_j \text{sign}(a_i - b_j), \]

Accordingly, relative strength $S(\beta, \alpha)$ is

\[S(\beta, \alpha) = \sum_j \sum_i \text{sign}(b_j - a_i), \]

The value of both relative strengths $S(\alpha, \beta)$ and $S(\beta, \alpha)$ can be represented by matrix $I$ \((m \times m)\) (see tb. 3).

The major diagonal of a matrix $I$ \((m \times m)\) is the vector-result of Blotto game between $\alpha$ and $\beta$ and trace of this matrix is payoff of Blotto game. Trace of the matrix tb. 3 equals 5, i.e. $\alpha$ wins $\beta$. Evidently, permutation of the matrix column will be change the matrix and possibly its trace. For example, if $\beta' = \{27, 11, 7, 9, 5, 3, 1\}$ the trace for such matrix equals $-1$, i.e. $\alpha$ loses $\beta$.

Numbering of some peculiarities of the matrices $I^*$ \((m \times 4)\) and $I$ \((m \times m)\):
Table 3: Matrix $I (7 \times 7)$ for $\alpha = \{21, 12, 10, 8, 6, 4, 2\}$ and $\beta = \{27, 11, 9, 7, 5, 3, 1\}$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>27</th>
<th>11</th>
<th>9</th>
<th>7</th>
<th>5</th>
<th>3</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>21</td>
<td></td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>12</td>
<td></td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
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<tr>
<td>10</td>
<td></td>
<td>-1</td>
<td>-1</td>
<td>1</td>
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<tr>
<td>8</td>
<td></td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
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<tr>
<td>6</td>
<td></td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
</tbody>
</table>

$\sum | -7 -3 -1 1 3 5 7 \quad S(\alpha, \beta) = 5$

- The values of the $S(\alpha, \beta)$ ($S^*(\alpha, \beta)$) do not depend on permutations of columns or rows of corresponding matrix $I$. Therefore, these values are invariants for Lotto game between $\alpha$ and $\beta$.
- For any $\alpha$ and $\beta$ from the set of $(n,m)$-partitions the bounds of relative strengths are

$$(2.9) \quad -(m^2 - m) \leq S(\alpha, \beta) \leq m^2 - m$$

For instance, these bounds holds for $(n,m)$-partitions $\alpha = \{n, 0, 0, \ldots, 0\}$ and $\beta = \{b_1, b_2, \ldots, b_m\}$, where $\forall b_i > 0$ and $\sum^m_i b_i = n$.

2.1 Two Partitions Features:
Balance and Peculiar Resource

To find of the useful for partitions ordering the numerical properties we introduced two characteristics connected with parts diversity and balance of partitions.

The $(n,m)$-partition has maximal diversity then and only when $a_1 > a_2 > \cdots > a_m > 0$, i.e. when the partition has all different parts.

Definition 2.3 Peculiar resource $S^*(\alpha, \alpha) \equiv S^*(\alpha)$ is

$$(2.10) \quad S^*(\alpha) = S^*(\alpha, \alpha) = \sum_{i=1}^{m} \sum_{j=1}^{m} \text{sign}^*(a_i - a_j).$$

where $\text{sign}^*(a_i - b_j) = 1$ if $a_i > b_j$ else $\text{sign}^*(a_i - b_j) = 0$.

There are next bounds for $S^*(\alpha)$

$$(2.11) \quad 0 \leq S^*(\alpha) \leq m(m - 1)/2,$$
The low bound holds for \((n, m)\)-partition if \(m/n\) is integer and all partitions parts are equals; upper bound — for strictly ordering sequential \(\alpha\). For example, in tb. 4 shown the matrix \(I\) \((4 \times 4)\) for \((16, 4)\)-partition with maximal value of peculiar resource \(S^*(\alpha) = 6\).

Table 4: Matrix \(I\) \((4 \times 4)\) for \(\alpha = \{7, 5, 3, 1\}\) with maximal value of peculiar resource.

<table>
<thead>
<tr>
<th>(\alpha)</th>
<th>7</th>
<th>5</th>
<th>3</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
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<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ \sum \begin{array}{cccc} 0 & 1 & 1 & 3 \\ \end{array} \]

\[ S^*(\alpha) = 6 \]

Unfortunately, known bounds (2.11) for numbers of partition with maximal peculiar resource is a rough for our purposes. So in what follows we will leans upon results of computing modeling for set of all \(6\,292\,069\) \((100, 10)\)-partitions (these partitions was generated by algorithm P in [10]).

In fig. 1 shown the distribution of the peculiar resource values for set of \((100, 10)\)-partitions. The maximal value of peculiar resource equals 45 has less than 0.2\% from all partition but “almost all” partitions has the values in range [38, 44]. Evidently algorithm to compute the value of peculiar resource for one pure strategy has the complexity \(\mathcal{O}(m^2 \log n)\).

![Figure 1: Histogram of the peculiar resource \(S^*\) for \((100, 10)\)-partitions (y-axis scale for values from 1 to 10 increased for clarity).](image)
such partition. For our goals more appropriate is definition of the permutation balance (PB) of partition.

**Definition 2.4** Permutation balance of \((n, m)\)-partition \(\alpha\) is

\[
PB(\alpha) = -\max \left| \frac{\sum_{i=1}^{m} a_i (i - m'_i)}{n} \right|,
\]

where

\[
m'_i = \begin{cases} 
(m + 1)/2 & \text{for odd } m \\
m/2 + 1 & \text{for even } m \text{ and } i \leq m/2 \\
m/2 & \text{for even } m \text{ and } i > m/2
\end{cases}
\]

and maximum is taken on the set of all \(m!\) permutations of \(\alpha\). (The value \(PB\) is defined by maximal deviation of the partition center mass from the partition middle. Therefore, value of \(PB\) the greater the less center mass deviates from the middle.)

For example, fig. 2 presented the distribution of all \((100, 10)\)-partitions accordance of the values permutation balance for each partition.

![Figure 2: Histogram of PB values for \((100, 10)\)-partitions.](image)

Let set all \((n, m)\)-partitions is \(\mathbb{R}(n, m)\). For PB values hold

\[
-m' \leq PB(\alpha) \leq 0, \quad \forall \alpha \in \mathbb{R}(n, m)
\]

\[
m' = \begin{cases} 
(m - 1)/2 & \text{for odd } m \\
m/2 & \text{for even } m
\end{cases}
\]

There upper bound holds for partition \(\{n/m, n/m, \ldots, n/m\}\), if \(n/m\) is integer and lower bound for partition \(\{n, 0, \ldots, 0\}\).

It is easy to verify that algorithmic complexity of PB computation does not exceed \(O(m^2 \log n)\) because PB(\(\alpha\)) reach the maximum when partition \(\alpha\) is partial ordering.
3 Approximation Results for the Lotto Game

Our aim is to design the approximation tractable procedures, which rated of the pure partition games strategies to accordance of their “winning ability”.

The exact decision of Lotto game for two \((n,m)\)-partitions \(\alpha\) and \(\beta\) is the triple real numbers \((\rho_w, \rho_l, \rho_t)\), where \(\rho_w\) is a fraction of wins \(\alpha\), \(\rho_l\) — a fraction of loses \(\alpha\) and \(\rho_t = 1 - (\rho_w + \rho_l)\) — a fraction of ties. Trivial algorithm to get the exact decision is computationally intractable even for \(m \geq 10\). Fortunately, for our aims it could be sufficiently to get of sign \((\rho_w - \rho_l)\) value only. What is more to compute this value sufficiently to compute of the potential resource or relative strengths values, i.e. the values \(S^*(\alpha, \beta)\) or \(S(\alpha, \beta)\). It is clear that one can compute these values with complexity \(O(m^2 \log n)\).

Let \(\alpha = \{a_1, a_2, \ldots, a_m\}\), \(\beta = \{b_1, b_2, \ldots, b_m\}\) and \(\Theta_\alpha\) is the set of all \(m!\) permutations of \(\alpha\). Denote as \(\Delta w(\alpha, \beta)\) and \(\Delta l(\alpha, \beta)\) part of expected values of the wins and losses, accordingly in game \(\alpha\) and \(\beta\) for the set \(\Theta_\alpha\).

**Proposition 3.1** Let \(\Gamma L(\alpha, \beta)\) is symmetric Lotto game with partitions \(\alpha = \{a_1, a_2, \ldots, a_m\}\) and \(\beta = \{b_1, b_2, \ldots, b_m\}\) and the payoff to \(\alpha\) against \(\beta\) is defined as (2.4). Then

1. If \(S^*(\alpha, \beta) \neq S^*(\beta, \alpha)\) for some \(\alpha\) and \(\beta\), then \(\text{sign} (\Delta w(\alpha, \beta) - \Delta l(\alpha, \beta)) = \text{sign} (S^*(\alpha, \beta) - S^*(\beta, \alpha))\).

2. There are such \(\alpha\) and \(\beta\) that \(S^*(\alpha, \beta) = S^*(\beta, \alpha)\) and \(\Delta w(\alpha, \beta) \neq \Delta l(\alpha, \beta)\).

See Appendix A for proof.

4 Tournaments: Experimental Results

To gain a better insight into the structure of the full set pure strategies for given \((n,m)\)-partitions we will simulated of the tournaments between all pure strategies of given parameters. It is simple (although very tedious) method. During these experiments we elucidate that there is strong dependencies the results from contest success function, i.e. what taken account of each pure strategy: cumulative payoff function value or number of wins.

The results for case \((36,6)\)-partitions when contest success function is cumulative payoff function shown in fig. 3.

In this case, partitions winning ability “monotone” grows with number of partition if these partitions were generated in reverse lexicographic order (starting with \(\{n,0,\ldots,0\}\)). Most “strength” partition is optimal balanced partition \(\{6,6,\ldots,6\}\).
However, the behavior of the tournament results is different when contest success function is cumulative number of wins for Lotto game (fig. 4).

It is clear from fig. 4 that winning ability of partitions unlike previous case fig. 3 not only grows with partitions numbers but begin to fall approximately after number 2000. The experiments above reinforce the statement in [5] that choice of playoff function is important for tournament results. However basic goal of our computing modelling was to get the answer on the next question: is the peculiar resource and permutation balance important for finding of the winners? In order to get the answer we conducted of the series experiments with (100, 10)-partitions. Firstly new definition be formulated

**Definition 4.1** Let $\mathbb{R}(n,m)$ is the set of all $(n,m)$-partitions and $I$ is the set of boundary conditions on $(n,m)$-partitions (for instance, the values of partitions balance).
Define as $\mathbb{R}(I, n, m) \subseteq \mathbb{R}(n, m)$ the subset of $(n, m)$-partitions $(\alpha_1, \ldots, \alpha_l)$ where each $\alpha_i \in \mathbb{R}(I, n, m)$.

Let $\mathbb{R}(I, n, m) \equiv C$ named class of partitions and $l$ — size of the $C$.

During all these experiments with different classes we considers two classes (teams) from $\mathbb{R}(n, m)$.

There have been modelling four tournaments for two teams each. First team is a subset $C_i$ of class $C_i$ ($i = 1, \ldots, 4$), where each class $C_i$ is given by two range of peculiar resource and permutation balance. Second team is $\mathcal{R}_i$ ($i = 1, \ldots, 4$) where $\mathcal{R}_j \subset \mathbb{R}(n, m)$.

Suppose that each round $t$ of tournament $i$ the teams $C_i$ and $\mathcal{R}_i$ play the pairwise match where players (pure strategies) picked by chance.

For all tournaments each partition from $C_i$ play Lotto game with each partition from $\mathcal{R}_i$. Thereupon the distributions of the fraction ”not lost” team $C_i$ against team $\mathcal{R}_i$ is computed.

Presented here are the results for next teams classes:

1. $C_1 \equiv \mathcal{R}_0 \subset \mathbb{R}(n, m)$ against $\mathcal{R}_1 \subset \mathbb{R}(n, m)$,
2. $C_2 \equiv \mathcal{S} \subset \mathcal{S}$ against $\mathcal{R}_2 \subset \mathbb{R}(n, m)$,
3. $C_3 \equiv \mathcal{E} \subset \mathcal{E}$ against $\mathcal{R}_3 \subset \mathbb{R}(n, m)$,
4. $C_4 \equiv \mathcal{SE} \subset \mathcal{SE}$ against $\mathcal{R}_4 \subset \mathbb{R}(n, m)$,

where $\mathcal{S}$, $\mathcal{E}$, $\mathcal{SE}$ are different subsets of $\mathbb{R}(n, m)$ with given values of the peculiar resource and permutation balance. The exact parameters of all subset and results of the tournaments (four first moments of the probabilistic distributions) are listed in table 5.

Table 5: Parameters of the teams and result of the tournaments.

<table>
<thead>
<tr>
<th>Team class $C_i$</th>
<th>Range of $S^*$</th>
<th>Total number of wins and ties</th>
<th>Moments of a fraction $\theta$ of the team $C_i$ in tournament with the team $\mathcal{R}_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{R}$</td>
<td>[0, 45]</td>
<td>6 292 069</td>
<td>$M[\theta] = 0.51444$, $D[\theta] = 0.07428$, $\mu_3 = -0.14220$, $\mu_4 = 1.83737$</td>
</tr>
<tr>
<td>$\mathcal{S}$</td>
<td>[43, 45]</td>
<td>2 290 362</td>
<td>$M[\theta] = 0.58334$, $D[\theta] = 0.05588$, $\mu_3 = -0.28689$, $\mu_4 = 2.02625$</td>
</tr>
<tr>
<td>$\mathcal{E}$</td>
<td>[0, 45]</td>
<td>9 165</td>
<td>$M[\theta] = 0.92685$, $D[\theta] = 0.00016$, $\mu_3 = -1.90790$, $\mu_4 = 9.06557$</td>
</tr>
<tr>
<td>$\mathcal{SE}$</td>
<td>[43, 45]</td>
<td>229</td>
<td>$M[\theta] = 0.93716$, $D[\theta] = 0.00002$, $\mu_3 = -1.20289$, $\mu_4 = 4.87540$</td>
</tr>
</tbody>
</table>

$^1$ The fraction of the total number of wins and ties to total number of loses for each partition of the team $C_i$.

It is much more informative to use a graphical plot for results from table 5 (see fig. 5–8). For all these graphics the abscissa is the value $(1 - \Delta l(\alpha, \beta))$ —
quantity of not lost games during all tournaments divided by total number of
the games for teams playing against \( R \), and ordinate is part of these teams
(\( R_0, S, \mathcal{E} \) or \( SE \), accordingly).

![Figure 5: Distribution of the fraction “not lost” partitions without any bound.](image)

![Figure 6: Distribution of the fraction “not lost” partitions with bound on peculiar resource.](image)

![Figure 7: Distribution of fraction “not lost” partitions with bound on partition balance](image)

From the above-mentioned results (tb. 5, fig. 5–8) it is reasonably safe to
suggest that peculiar resource and partition balance are reliable indicated of
pure strategies winning ability. Indeed if one choose the set of pure strategies
\( C(n, m) \) with some predefined range of these parameters one might expect
that \( C(n, m) \) wins the tournament against “almost” any set of pure strategies
\( R(n, m) \), i. e. expected value of the fractions wins for \( C(n, m) \) would be strongly
more 1/2 at least.
Discussions

Over last few years researches in the field of the algorithmic nature of the game theory grows exponentially. Not only algorithm for computing equilibrium (in particularly, $\epsilon$-approximate equilibrium\(^1\)) have been researches goal but (and may be foremost) the problems of the validity of behavior prediction. Efficient (or tractable) computability has emerged as a very desirable feature of such predictions. Besides design of the game (see for reference for example, [6]), especially in the case of auctions has become the arena of intense game theory and algorithm researches.

However, it seems that antagonistic constant-sum matrix games could be important goal for algorithmic approaches also. Allocative games such as traditional Blotto and Lotto games should first of all be studied. The reasons are (besides wide ranges of practical applications these games) behavioral problems connected with these games. In [1] the results of the massive experiment to understand as people to decide of Colonel Blotto game ((120, 6)-partitions) was conducted. It turned out that none of gamers play equilibrium but all used some heuristic reasoning (by the way, winning ability for top-10 results was not very high – more than half fields wins about 10% participants from 6 000 only and maximal score was approximately 0.63).

Our computer modelling pointed the way for conducting experimental researches to establish people behavior in cases of allocative games. Seems choosing of the games parameters ($n$ and $m$) is significantly and may be more important get ordering all pure strategies by first. In this case only one can be inferred how successful people reasoning and why these reasoning lost.

The second line of future investigation in the field experimental studying of partition games is adjacent to the fictitious plays (FP) (see [4] for list of reference). It is common knowledge that FP proceeds in rounds. In the first round, each player arbitrarily chooses one of his actions and in next rounds,

\(^1\)For example, in [14] give the definition of $\epsilon$-approximate Nash equilibrium as a mixed strategy profile such that no other strategy can improve the payoff by more than an addition $\epsilon$ ($\epsilon > 0$).
each player try to find empirical frequency of play of their opponents in previous rounds, to plays a pure best response strategy. Therefore, FP can be seen as a learning algorithm for real repeated games. Experimental studying of partition game could be useful for practical estimates such important features learning algorithm as speed converge to approximate equilibrium and choose of the effective actions during learning.

Next important direction of the future investigation is researches in the field evolution game (see, for example, [17]). Our experiments shows that model of evolutionary game dynamics where actors are the partitions and the rule of game are identical to that for a Lotto game demonstrated some interesting results of evolutionary game dynamics among them the simple condition of appearance evolutionary stability strategies (ESS) for different replication laws.

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References


**Appendix A  Proof of Proposition 3.1**

First we will used result in [8] which statement that when Colonel Lotto game and Colonel Blotto game has the same parameters $n$ and $m$ and each mixed strategies Lotto game has probability $1/m$ to each one of the $m$ permutation of the pure strategy then these games have the same value. Moreover, the set
of the optimal strategies in the Colonel Lotto game mapping onto the optimal strategies in the Colonel Blotto game.

Notes that for any $I$ matrix for symmetrical game of two $(n,m)$-partitions $\alpha = \{a_1, a_2, \ldots, a_m\}$ and $\beta = \{b_1, b_2, \ldots, b_m\}$ there is injection $I^*$ matrix unsymmetrical game of two vectors of the potential resource $R^*(\alpha, \beta)$ and $R^*(\beta, \alpha)$ (see (2.6)). Besides the values of the $S^*(\alpha, \beta)$, $S^*(\beta, \alpha)$ (see (2.5)) does not depend on permutations of the columns or rows an accordingly matrix.

Let the resource of strong player is $R_s$ and weak player is $R_w$. Suppose that strong player and weak player allocate resources in each front based on the univariate distribution functions from [16] and resources in each front randomly chosen from $[0, 2 \cdot \text{(resource player)}/m]$.

Then in accordance [16], the expected proportions of battlefields wins by weak player is $\Delta_w = \frac{R_w}{2R_s}$ and for strong player is $\Delta_s = 1 - \Delta_w$.

Condition for resource maximal values for each front is true always because maximal potential resource does not exceed $(m - 1)^2$ and maximal resource of each front does not exceed $m$.

We shall deal with two cases for the relation $R_w/R_s$ from [16].

a) $2/m \leq R_w/R_s \leq 1$

b) $1/(m - 1) \leq R_w/R_s \leq 2/m$

Let for potential resources $S^*(\alpha, \beta) > S^*(\beta, \alpha)$, i.e. $\alpha$ is a strong player and $\beta$ is a weak player in terminology [16]. Because for potential resources one have estimate

$$S^*(\alpha, \beta) + S^*(\beta, \alpha) \leq m^2$$

the case a) does not true only when $S^*(\alpha, \beta) = (m - 1)^2$ and $S^*(\beta, \alpha) = m$. However, for such strategies case b) is true.

Therefore for the case a) expected fraction of winning fronts for weak player is

$$\Delta_w = S^*(\beta, \alpha)/(2 \cdot S^*(\alpha, \beta))$$

and for strong player is $\Delta_s = 1 - \Delta_w$.

For case b)

$$\Delta_w = 2/m - (2 \cdot S^*(\beta, \alpha))/(m^2S^*(\alpha, \beta)),$$

$$\Delta_s = 1 - \Delta_w.$$

It means that for both cases if $S^*(\alpha, \beta) > S^*(\beta, \alpha)$ then $\Delta_w(\alpha, \beta) > \Delta_w(\beta, \alpha)$.

In other words mathematical expectation of the fraction winning fronts for Lotto game would be strictly more (strictly less) for such pure strategy whose potential resource is strictly more (strictly less) than rival’s potential resource.
To prove two part of Proposition 3.1 sufficiently demonstrated at least one example on table 6. Here shown $I^*$ matrix for two partition $\alpha = \{4, 2, 2, 0\}$ and $\beta = \{5, 3, 0, 0\}$. For these partitions $S^*(\alpha, \beta) = S^*(\beta, \alpha) = 7$, but fraction $\Delta w(\alpha, \beta) = 3/12$ does not equal fraction $\Delta l(\alpha, \beta) = 4/12$.

Table 6: $I^*$ matrix for $\alpha = \{4, 2, 2, 0\}$ and $\beta = \{5, 3, 0, 0\}$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>4 2 2 0</th>
<th>$S^*(\alpha, \beta) = 7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R^*(\alpha, \beta)$</td>
<td>3 2 2 0</td>
<td></td>
</tr>
<tr>
<td>$\beta$</td>
<td>5 3 0 0</td>
<td>$S^*(\beta, \alpha) = 7$</td>
</tr>
<tr>
<td>$R^*(\beta, \alpha)$</td>
<td>4 3 0 0</td>
<td></td>
</tr>
</tbody>
</table>


Table 7: Total number of such pairs $(\alpha, \beta)$ from the set of all $(n, m)$-partitions that $S^*(\alpha, \beta) = S^*(\beta, \alpha)$ and $\Delta w(\alpha, \beta) = \Delta l(\alpha, \beta)$ or $\Delta w(\alpha, \beta) \neq \Delta l(\alpha, \beta)$. The order of partition in a pair is ignored, i. e. $(\alpha, \beta)$ and $(\beta, \alpha)$ are the same for particular computations.

<table>
<thead>
<tr>
<th>$n \ m$</th>
<th>$\Delta w \neq \Delta l$</th>
<th>$\Delta w = \Delta l$</th>
<th>Total number of pairs</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 2</td>
<td>0</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>6 3</td>
<td>0</td>
<td>7</td>
<td>21</td>
</tr>
<tr>
<td>9 3</td>
<td>0</td>
<td>21</td>
<td>66</td>
</tr>
<tr>
<td>8 4</td>
<td>1</td>
<td>27</td>
<td>105</td>
</tr>
<tr>
<td>12 4</td>
<td>12</td>
<td>118</td>
<td>561</td>
</tr>
<tr>
<td>16 4</td>
<td>50</td>
<td>386</td>
<td>2 016</td>
</tr>
<tr>
<td>10 5</td>
<td>3</td>
<td>52</td>
<td>435</td>
</tr>
<tr>
<td>15 5</td>
<td>70</td>
<td>308</td>
<td>3 486</td>
</tr>
<tr>
<td>25 5</td>
<td>2 209</td>
<td>4 672</td>
<td>70 876</td>
</tr>
<tr>
<td>12 6</td>
<td>25</td>
<td>179</td>
<td>1 653</td>
</tr>
<tr>
<td>18 6</td>
<td>360</td>
<td>1 535</td>
<td>19 701</td>
</tr>
<tr>
<td>24 6</td>
<td>2 789</td>
<td>9 687</td>
<td>141 246</td>
</tr>
<tr>
<td>30 6</td>
<td>14 483</td>
<td>47 648</td>
<td>726 615</td>
</tr>
<tr>
<td>36 6</td>
<td>59 393</td>
<td>191 735</td>
<td>2 956 096</td>
</tr>
</tbody>
</table>

Note. Probably Proposition 3.1.1 holds for exact values of the wins or losses not only for their expected values. At least our numerical experiments testifies that. But the issue remains open.

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