A Generalized Method for
Exact Solutions to NLPDE’s

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Abstract

Using the idea of the very recently generalized Kudryashov method and based on a generalized Riccati equation we present an alternative method which can be considered as an extension of the well known tanh method. As application of the proposed method, we obtain exact solutions of the Dispersive modified Benjamin-Bona-Mahony equation showing in this way its effectiveness.

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1 Introduction

As we know, the implementation of new methods for the search of exact solutions for (NLPDE’s) is a relevant task. Due to the availability of computer symbolic systems like Maple or Mathematica the task of finding computational methods to obtain exact solutions to nonlinear partial differential equations is very attractive for many researches. One of the most popular methods for obtain exact solutions for several NLPDE’s that appear in the mathematical physics is the tanh method [1]. The tanh method is very similar to the more
recently Kudryashov method [2], the difference consists on the calculations of
the derivatives. However, both methods have been used widely for solving
many nonlinear partial differential equations (NLPDE’s) in a satisfactory way
[1],[2]. Recently, a generalized Kudryashov method [3] have been implemented
and it have been applied for to solve many NLPDE’s. Using the idea of this last
method, we present an alternative method which can be considered as more
general that both, the tanh method and the generalized Kudryashov method.
With the aim to show the effectiveness of the method, we use it to obtain
exact solution of the following Dispersive modified Benajmin-Bona-Mahony

\[ u_t(x,t) + u_x(x,t) - \delta u^2(x,t)u_x(x,t) + u_{xxx} = 0, \]  \tag{1}  

with \( \delta \) a nonzero constant. The equation (1) have relevance in the physic of
waves [4]. The paper is organized as follows: In Sec. 2, we give a description
of the method. In Sec. 3, we use the method for obtain exact solutions to (1).
Finally, some conclusions are given.

2 Description of the method

Suppose \( u \) is a differentiable function in the variables \( x \) and \( t \). For a given
NLPDE that does not explicitly involve independent variables

\[ P(u, u_t, u_x, u_{xt}, u_{xx}, ... ) = 0, \]  \tag{2}  

we use the wave transformation

\[ \xi = x + \lambda t + \xi_0, \]  \tag{3}  

where \( \lambda \) is the speed of the wave and \( \xi_0 \) is an arbitrary constant. Substituting
(3) into (2) we obtain an ordinary differential equation in the variable \( \xi \)

\[ P_1(u, u', u'', ... ) = 0, \]  \tag{4}  

where \( ' \) denote the derivative with respect to the new variable \( \xi \). In the next
step, we consider the solution of (4) in the form

\[ u(\xi) = \frac{\sum_{i=0}^{n} a_i \phi(\xi)^i}{\sum_{k=0}^{m} b_k \phi(\xi)^k}, \]  \tag{5}  

where \( n \) and \( m \) are integers to be determinate balancing the linear term of
the highest order with the nonlinear term in (5). The new variable \( \phi(\xi) \) is the
solution of the generalized Riccati equation

\[ \phi'(\xi) = \alpha + \beta \phi(\xi) + \gamma \phi^2(\xi). \]  \tag{6}
The solution of (6) is given by [5]

$$
\phi(\xi) = \left\{ \begin{array}{ll}
-\sqrt{\beta^2 - 4\alpha\gamma} \tan h \left[ \frac{1}{2} \sqrt{\beta^2 - 4\alpha\gamma}\xi \right] + \beta & \text{if, } \beta^2 - 4\alpha\gamma \neq 0 \\
-\frac{1}{\gamma} \left[ a + \frac{\beta}{2} \right], & \text{if, } \beta^2 - 4\alpha\gamma = 0
\end{array} \right. 
$$

Substituting (5) into (4), taking into account (6) and after simplification we obtain a rational expression. Equating to zero all the coefficients of $\phi_i(\xi)$ in the numerator of this fraction, we obtain a set of algebraic equations in the unknowns $a_i$, $i = 1, \ldots, a_n$, $b_k$, $k = 1, \ldots, m$, $\lambda$, $\alpha$, $\beta$, and $\gamma$. Solving this system with aid of symbolic computation and reversing the transformations we obtain the exact solution to (2).

If we have $b_i = 0$ for $i = 1, 2, \ldots, b_m$ we obtain a generalized tanh method. In the case $\alpha = 0$, $\beta = -1$ and $\gamma = 1$ we have the generalized Kudryashov method [3]. Moreover, some solutions obtained by the extended tanh-coth method [6] can be derived with this method.

## 3 Exact solution for equation (1)

Applying (3) to (1) and after one integration, we have

$$
3(1 + \lambda)u(\xi) - \delta u^3(\xi) + 3u''(\xi) + k_1 = 0,
$$

where $k_1$ is the integration constant. Substituting (5) into (8) and balancing $u^3(\xi)$ with $u''(\xi)$ we have $3n - 3m = n - m + 2$ and therefore we obtain the relation $n = m + 1$. For sake of simplicity we consider $m = 1$ so that $n = 2$, and then (5) take the form

$$
u(\xi) = \frac{a_0 + a_1\phi(\xi) + a_2\phi^2(\xi)}{b_0 + b_1\phi(\xi)}.
$$

Substituting (9) into (8) and after simplifications we have a rational expression. Equating to zero the coefficients of $\phi^i(\xi)$ for $i = 0, 1, 2, \ldots, 6$ in the numerator we have a system of seven algebraic equations. Solving it respect to the unknowns $a_0$, $a_1$, $a_2$, $b_0$, $b_1$, $\lambda$, $\alpha$, $\beta$, and $\gamma$ we obtain the following general solution:
\[
\begin{align*}
a_0 &= 0 \\
a_1 &= \frac{3(-1)^{2/3}b_1 \sqrt[3]{k_1}}{2^{2/3} \sqrt[3]{\delta}} \\
a_2 &= \frac{(-1)^{2/3} \sqrt[3]{3} k_1}{b_0 \sqrt[3]{\delta}} \\
\alpha &= \frac{(-1)^{2/3} b_0 \sqrt[3]{3} \sqrt[3]{k_1}}{2 \sqrt[3]{2} \sqrt[3]{3} \sqrt[3]{\delta}} \\
\beta &= \frac{(-1)^{2/3} \sqrt[3]{3} \sqrt[3]{k_1}}{b_0 \sqrt[3]{3} \sqrt[3]{\delta}} \\
\gamma &= \frac{(-1)^{2/3} b_0 \sqrt[3]{3} \sqrt[3]{k_1}}{2 \sqrt[3]{2} \sqrt[3]{3}} \\
\lambda &= \frac{1}{4} \left( \sqrt{-1} 2^{2/3} \sqrt[3]{3} \sqrt[3]{k_1}^{2/3} - 4 \right) .
\end{align*}
\]

In accordance with (7) we have

\[
\phi(\xi) = -\sqrt{-1} \sqrt[3]{3} b_0 \left( \frac{-(-1)^{2/3} \sqrt[3]{3} \sqrt[3]{k_1}}{\sqrt[3]{2} \sqrt[3]{3}} - \sqrt{-\frac{\sqrt{3}}{2} \sqrt[3]{\delta} k_1^{2/3}} \tanh \left( \frac{1}{2} \sqrt{-\frac{\sqrt{3}}{2} \sqrt[3]{\delta} k_1^{2/3}} \xi \right) \right) .
\]

Finally, taking into account (11), (9) and (3), the solution to (1) respect to (10) is given by

\[
u(x,t) = \frac{3(-1)^{2/3} b_1 \sqrt[3]{k_1}}{2^{2/3} \sqrt[3]{\delta}} \phi(\xi) + \frac{(-1)^{2/3} \sqrt[3]{3} \sqrt[3]{k_1}}{b_0 \sqrt[3]{3} \sqrt[3]{\delta}} \phi^2(\xi),
\]

where \( \phi(\xi) \) is given by (11), \( \xi = x + \frac{1}{4} \left( \sqrt{-1} 2^{2/3} \sqrt[3]{3} \sqrt[3]{k_1}^{2/3} - 4 \right) t + \xi_0, b_0, b_1, k_1 \) and \( \xi_0 \) are arbitrary constants.

4 Conclusions

We have presented a generalized method for obtain traveling wave solutions for NLPDE’s. The standard tanh method, the generalized tanh method, the Kudryashov method and the generalized Kudriashov method can be derived as particular cases. The method have used to obtain exact solution of the Dispersive modified Benjamin-Bona-Mahony equation, showing in this way the effectiveness.

References


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