A Buffon-Laplace Type Problem for an Irregular Lattice and Different Obstacles

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Abstract

In the previous papers, [2], [3], [4], [5], [6], [7], [8], [9], [10], the authors studies some Laplace problem for different lattices and different obstacles. In this paper we consider two regular lattices with the cell represented as in figure 1 and we compute the probability that a random segment of constant length intersects a side of lattice. In particular we obtain the probability determinated in the previous work, then the Laplace probability.

Keywords: Geometric Probability, stochastic geometry, random sets, random convex sets and integral geometry

1 Introduction

In 1773, in according of a meeting of the Académic des Sciences de Paris, Buffon posed a problem that later on should become knows as the famous “Buffon needle problem”: in a room, the floor of which is merely divided by parallel lines, at a distance a apart, a needle of length $l < a$ is allowed to fall at random: which is the probability that the needle intersects one of the lines? The solution, determined by Buffon by means of empirical methods,
was \( p = \frac{2l}{\pi a} \). The problem and its solution were published in 1777, in the Comptes rendus de l’Académie des sciences de Paris”.

In 1812, Laplace extended the problem by considering a room paved with equal tiles, shaped as rectangles of sides \( a \) and \( b \), with \( l < \min(a, b) \). The solution was \( p = \frac{2l(a+b)-l^2}{\pi ab} \), and it is obvious that the probability of Buffon can be obtained from that of Laplace by letting \( b \to +\infty \).

This is his first work on the needle problem. It was ignored for a long time until it was rediscovered in 1869 by the English mathematician W. Morgan Crofton, which will have a central role in the structuring of geometric probability, in which, to simplify, the calculation of the probability of a certain event has a geometric interpretation, in the sense that the set of possible events will have as image a certain figure \( F \) of the representational space, while the set of events will be favorable to picture a figure \( F_1 \) contained in \( F \), and these figures will have to make the measurement.

K. Baclawski, M. Cerasoli and G. C Rota, in their book “Introduction to Probability” [1], have found that Buffon’s needle problem has done so much to discuss the mathematicians of the nineteenth century due to the difficulty of making rigorous, and because it involves the presence of probability. In fact, it gives a technique by which you might find probabilistically (as was done) an approximate value of \( p \).

In particular, from 1974, several authors have shown different and innovative characterizations to this type of problem, they have considered several extensions in different directions: other types of regular and irregular lattices; other type test bodies; other spaces with dimension higher that two.

We restate now these problems in a slightly different form, wide will be useful for several different extensions.: Let \( R \) be a lattice in the plane, with elementary cell \( C \), and let \( T \) be a compact convex set, with fixed shape and dimensions, but random position in the plane (test body). As a natural assumption of randomness, we consider \( T \) as uniformly distributed in a bounded region of the plane. We also suppose that there exists at least one position of \( T \) such that \( T \) is entirely contained in \( C \).

The problem is to find which is the probability that the body \( T \) intersects the boundary of the cells \( C \) of the lattice \( R \).

In Buffon’s needle problem the cell \( C \) is a strip of breadth \( a \), in the problem of Laplace \( C \) is a rectangle of sides \( a \) and \( b \) and in both cases \( T \) is a segment of length \( l \). The study of Buffon-Laplace problems for lattices with obstacles is of interest in many fields, as such lattices are useful in modelling samples of biological tissue, in the geometric distribution of vegetation, in transport and problem of traffic, in geological structures, in crystallography, in problems of quality in porous materials, in topography and in many other fields.

Now we consider a Laplace type problem for a fundamental cell composed by four isosceles triangles considering as body test a rectangle.
2 Main result

Let $\mathcal{R}(a;m)$ the lattice with the fundamental cell $C_0$ rappresented in fig. 1

\[ |DF| = \frac{a}{2 \cos \frac{\pi}{5}}, \quad |B_1B_4| = |D_1D_2| = |E_2E_3| = m \cos \frac{\pi}{5}, \]

\[ |D_2D_3| = 2m \sin \frac{\pi}{5}, \quad \] (1)

\[ \text{area}AA_1A_4 = \text{area}CC_1C_4 = \frac{3m^2}{40}, \quad \text{area}DD_2D_3 = \frac{\pi m^2}{20}, \]

\[ \text{area}EE_1E_2 = \frac{m^2}{8} \sin \frac{\pi}{5}, \]

\[ \text{area}BB_1B_4 = \text{area}DD_1D_2 = \text{area}EE_2E_3 = \text{area}FF_1F_2 = \frac{m^2}{8} \sin \frac{2\pi}{5}. \quad (2) \]

Hence we have

\[ \text{area}C_{01} = \frac{a^2}{2} \sin \frac{\pi}{5} \left( 1 + 2 \cos \frac{\pi}{5} \right) = \frac{3m^2}{8} \left( \sin \frac{2\pi}{5} + \frac{6\pi}{5} \right), \quad (3) \]
\[ \text{area}_C = \frac{\alpha^2}{2} \sin \frac{\pi}{5} - m^2 \left( \sin \frac{2\pi}{5} + \sin \frac{\pi}{5} + \frac{2\pi}{5} \right). \] (4)

\[ \text{area}_{C_0} = a^2 \sin \frac{\pi}{5} \left( 1 + \cos \frac{\pi}{5} \right) - m^2 \left( 4 \sin \frac{2\pi}{5} + \sin \frac{\pi}{5} + 4\pi \right). \] (5)

We want to compute the probability that a random segment \( s \) of constant length \( l < \frac{a}{4\cos \frac{\pi}{5}} - \frac{m}{2} \) intersects a side of the lattice, i.e. the probability \( P_{\text{int}} \) that \( s \) intersects a side of the fundamental cell \( C_0 \).

The position of the segment \( s \) is determined by center and by the angle \( \varphi \) that \( s \) formed with the line \( CE \).

In order to compute \( P_{\text{int}} \) we consider the limiting positions of segment \( s \) in the cell \( C_0 \).

and the formulas

\[ \text{area}_{\hat{C}_{01}} (\varphi) = \text{area}_C - \sum_{i=1}^{10} \text{area}_{a_i} (\varphi), \] (6)

\[ \text{area}_{\hat{C}_{02}} (\varphi) = \text{area}_C - \sum_{i=1}^{6} \text{area}_{b_i} (\varphi). \] (7)

We have that

\[ \text{area}_{a_1} (\varphi) = \frac{lm}{2} \cos \frac{\pi}{5} \sin \varphi - m^2 \left( \frac{3\pi}{5} - \sin \frac{2\pi}{5} \right), \]
Buffon-Laplace type problem

\[ \text{area}_{a_3}(\varphi) = \frac{l^2}{2} \sin\left(\varphi - \frac{\pi}{5}\right) \sin\left(\frac{2\pi}{5} - \varphi\right), \]

\[ \text{area}_{a_2}(\varphi) = \frac{(a - m)l}{2} \sin \left(\varphi - \frac{\pi}{5}\right) - \]

\[ \frac{l^2}{2 \sin \frac{\pi}{5}} \sin \left(\varphi - \frac{\pi}{5}\right) \sin \left(\frac{2\pi}{5} - \varphi\right). \]

The fig. 2 give us \( a_1 = a_5 \), then

\[ \text{area}_{a_5}(\varphi) + \text{area}_{a_1}(\varphi) = \frac{lm}{2} \cos \frac{\pi}{5} \sin \varphi - \frac{m^2}{8} \left(\frac{3\pi}{5} - \sin \frac{2\pi}{5}\right), \]

\[ \text{area}_{a_4}(\varphi) = \frac{(a - m)l}{2} \sin \left(\frac{2\pi}{5} + \varphi\right), \]

\[ \text{area}_{a_6}(\varphi) = \frac{(a - m)l}{2} \sin \varphi, \]

\[ \text{area}_{a_7}(\varphi) = \frac{l^2}{2 \sin \frac{\pi}{5}} \sin \left(\varphi - \frac{\pi}{5}\right) \sin \left(\frac{2\pi}{5} - \varphi\right), \]

\[ \text{area}_{a_8}(\varphi) = \frac{(\alpha - m)l}{2} \sin \left(\frac{2\pi}{5} - \varphi\right) - \]

\[ \frac{l^2}{2 \sin \frac{\pi}{5}} \sin \left(\varphi - \frac{\pi}{5}\right) \sin \left(\frac{2\pi}{5} - \varphi\right), \]

\[ \text{area}_{a_9}(\varphi) = \frac{ml}{2} \cos \frac{\pi}{5} \sin \left(\frac{2\pi}{5} + \varphi\right), \]

\[ \text{area}_{a_{10}}(\varphi) = \frac{(a - m)l}{2} \sin \left(\varphi + \frac{\pi}{5}\right), \]

Then we obtain

\[ \text{area}_{\hat{C}_{01}}(\varphi) = \text{area}_{C_{01}} - \left\{ al \left[ \sin \frac{2\pi}{5} \cos \varphi + \left(\cos \frac{\pi}{5} + \frac{1}{2}\right) \sin \varphi \right] + \right. \]

\[ \left. \frac{lm}{2} \left[ \sin \frac{\pi}{5} \cos \frac{2\pi}{5} - 2 \cos \frac{\pi}{5} \right] \cos \varphi + \left(\cos \frac{\pi}{5} \cos \frac{2\pi}{5} + \cos \frac{\pi}{5} - 1\right) \sin \varphi \right\} + \]
\[
\frac{l^2}{2 \cos \frac{2\pi}{5}} \left( \cos \frac{2\pi}{5} \cos 2\varphi + \sin \frac{2\pi}{5} \sin 2\varphi - 1 \right) - \frac{m^2}{4} \left( \frac{3\pi}{5} - \sin \frac{2\pi}{5} \right) \right) .
\]

Denoting
\[A_1(\varphi) = \sum_{i=1}^{10} \text{area}_{a_i}(\varphi) ,\]
the last can be write
\[
\text{area} \hat{C}_{01}(\varphi) = \text{area} C_{01} - A_1(\varphi) ,
\]
where
\[A_1(\varphi) = a l \left[ \sin \frac{2\pi}{5} \cos \varphi + \left( \cos \frac{\pi}{5} + \frac{1}{2} \right) \sin \varphi \right] + \]
\[
\frac{lm}{2} \left[ \sin \frac{\pi}{5} \left( \cos \frac{2\pi}{5} - 2 \cos \frac{\pi}{5} \right) \cos \varphi + \left( \cos \frac{\pi}{5} \cos \frac{2\pi}{5} + \cos \frac{\pi}{5} - 1 \right) \sin \varphi \right] + \]
\[
\frac{l^2}{2 \cos \frac{2\pi}{5}} \left( \cos \frac{2\pi}{5} \cos 2\varphi + \sin \frac{2\pi}{5} \sin 2\varphi - 1 \right) - \frac{m^2}{4} \left( \frac{3\pi}{5} - \sin \frac{2\pi}{5} \right) .
\]

To compute \(\text{area} \hat{C}_{02}(\varphi)\) we have that:
\[\text{areab}_1(\varphi) = \frac{l^2}{2 \sin \frac{\pi}{5}} \sin \left( \frac{2\pi}{5} - \varphi \right) \sin \left( \frac{2\pi}{5} + \varphi \right) - \frac{m^2}{8} \sin \frac{\pi}{5},\]
\[\text{areab}_2(\varphi) = \frac{(2a - m) l}{4} \sin \left( \frac{2\pi}{5} - \varphi \right) - \]
\[
\frac{l^2}{2 \sin \frac{\pi}{5}} \sin \left( \frac{2\pi}{5} - \varphi \right) \sin \left( \frac{2\pi}{5} + \varphi \right) ,
\]
\[\text{areab}_3(\varphi) = \frac{l^2}{4} \cos \left( \varphi - \frac{\pi}{5} \right) - \frac{m^2}{8} \left( \frac{2\pi}{5} - \sin \frac{2\pi}{5} \right) ,
\]
\[\text{areab}_5(\varphi) = \frac{l^2 \sin \varphi \cos \left( \varphi - \frac{\pi}{5} \right)}{2 \sin \frac{2\pi}{5}} - \frac{m^2}{8} \sin \frac{2\pi}{5} ,
\]
\[\text{areab}_4(\varphi) = \left( \frac{a}{2 \cos \frac{\pi}{5}} - \frac{m}{2} \right) \frac{l}{2} \sin \varphi - \frac{l^2 \sin \varphi \cos \left( \varphi - \frac{\pi}{5} \right)}{2 \sin \frac{2\pi}{5}} ,
\]
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\[ \text{area}_{\theta_0} (\varphi) = \frac{a l}{2} \sin \left( \frac{2\pi}{5} + \varphi \right) - \frac{l^2}{2 \sin \frac{2\pi}{5}} \sin \left( \frac{2\pi}{5} - \varphi \right) \sin \left( \frac{2\pi}{5} + \varphi \right) - \]
\[ \frac{l^2}{2 \sin \frac{2\pi}{5}} \sin \varphi \sin \left( \frac{2\pi}{5} + \varphi \right). \]

Then we obtain that

\[ \text{area}_{\hat{C}_0_2} (\varphi) = \text{area}_{C_0_2} - \left\{ al \left( \sin \frac{2\pi}{5} \cos \varphi + \frac{1}{4} \cos \frac{\pi}{5} \sin \varphi \right) - \right. \]
\[ \frac{ml}{4} \left[ \sin \left( \frac{2\pi}{5} - \varphi \right) + \sin \varphi \right] + \frac{l^2}{4} \left[ \cos \left( \varphi - \frac{\pi}{5} \right) + \left( \cot \frac{2\pi}{5} - \frac{1}{\sin \frac{\pi}{5}} \right) \cos 2\varphi - \right. \]
\[ \sin 2\varphi + \cos \left( \varphi - \frac{\pi}{5} \right) - \frac{1}{\sin \frac{\pi}{5}} - \cot \frac{2\pi}{5} \right] - \frac{m^2}{8} \left( \sin \frac{\pi}{5} + \frac{2\pi}{5} \right) \}. \]

(10)

Denoting

\[ A_2 (\varphi) = al \left( \sin \frac{2\pi}{5} \cos \varphi + \frac{1}{4} \cos \frac{\pi}{5} \sin \varphi \right) - \]
\[ \frac{ml}{4} \left[ \sin \left( \frac{2\pi}{5} - \varphi \right) + \sin \varphi \right] + \frac{l^2}{4} \left[ \left( \cot \frac{2\pi}{5} - \frac{1}{\sin \frac{\pi}{5}} \right) \cos 2\varphi - \right. \]
\[ \sin 2\varphi + \cos \left( \varphi - \frac{\pi}{5} \right) - \frac{1}{\sin \frac{\pi}{5}} - \cot \frac{2\pi}{5} \right] - \frac{m^2}{8} \left( \sin \frac{\pi}{5} + \frac{2\pi}{5} \right), \]

(11)

the formula (10) became

\[ \text{area}_{\hat{C}_0_2} (\varphi) = \text{area}_{C_0_2} - A_2 (\varphi). \]

(12)

By the formulas (9) and (11) follows

\[ A_1 (\varphi) + A_2 (\varphi) = al \left[ 2 \sin \frac{2\pi}{5} \cos \varphi + \left( \cos \frac{\pi}{5} + \frac{1}{2} + \frac{1}{4 \cos \frac{\pi}{5}} \right) \sin \varphi \right] + \]
\[ \frac{lm}{2} \left[ \sin \frac{\pi}{5} \cos \frac{2\pi}{5} \cos \varphi + \left( \cos \frac{2\pi}{5} \cos \frac{\pi}{5} + \cos \frac{2\pi}{5} + \cos \frac{\pi}{5} - 2 \right) \sin \varphi \right] + \]
\[
\frac{l^2}{4} \left[ \left( \frac{2\pi}{5} \cdot \cot \frac{\pi}{5} - \frac{1}{\sin \frac{\pi}{5}} + 2 \right) \cos 2\varphi + \left( \frac{2\tan \frac{2\pi}{5} - 1}{\cos \frac{2\pi}{5} - \frac{1}{\sin \frac{\pi}{5}}} \right) \cos \left( \varphi - \frac{\pi}{5} \right) - \frac{m^2}{8} \left( \frac{8\pi}{5} + \sin \frac{\pi}{5} - \frac{2\sin \frac{2\pi}{5}}{5} \right) \right]
\]

(13)

Denoting with \( M_i \) \((i = 1, 2)\), the set of the segment \( s \) that have the middle point in the cell \( C_{0i} \) and with \( N_i \) the set of the segment \( s \) completely in \( C_{0i} \) we have [12]

\[
P_{\text{int}} = 1 - \frac{\mu (N_1) + \mu (N_2)}{\mu (M_1) + \mu (M_2)},
\]

(14)

where \( \mu \) is the Lebesgue measure in the euclidean plane.

To compute the measure \( \mu (N_i) \) and \( \mu (M_i) \) we use the kinematic measure of Poincarè [11]:

\[
dk = dx \wedge dy \wedge d\varphi,
\]

where \( x, y \) are the coordinate of middle point of segment \( s \) and \( \varphi \) the fixed angle.

We have that:

\[
\mu (M_i) = \int_{\pi/5}^{2\pi/5} d\varphi \int_{\{(x,y) \in C_{0i}\}} dx dy = \int_{\pi/5}^{2\pi/5} (\text{area } C_{0i}) \ d\varphi = \frac{\pi}{5} \text{area } C_{0i},
\]

then

\[
\mu (M_1) + \mu (M_2) = \frac{\pi}{5} \text{area } C_0,
\]

(15)

and

\[
\mu (N_i) = \int_{\pi/5}^{2\pi/5} d\varphi \int_{\{(x,y) \in \hat{C}_{0i}(\varphi)\}} dx dy = \int_{\pi/5}^{2\pi/5} \left[ \text{area } \hat{C}_{0i} (\varphi) \right] dy =
\]

\[
\frac{\pi}{5} \text{area } C_{0i} - \int_{\pi/5}^{2\pi/5} \left[ A_i (\varphi) \right] d\varphi,
\]
Then,

\[ P_{\text{int}} = \frac{5}{\pi \text{area} C_0} \int_{\pi/5}^{2\pi/5} \left[ A_1(\varphi) + A_2(\varphi) \right] d\varphi. \]  \tag{16} 

Considering that

\[ \int_{\pi/5}^{2\pi/5} \left[ A_1(\varphi) + A_2(\varphi) \right] d\varphi = \]

\[ al \left( \frac{1}{2} + \cos \frac{\pi}{5} - \frac{1}{4 \cos \frac{\pi}{5}} \right) \left( \cos \frac{2\pi}{5} - \cos \frac{\pi}{5} \right) + \frac{lm}{2} \left( \cos \frac{\pi}{5} - \cos \frac{2\pi}{5} \right) + \]

\[ \frac{l^2}{4} \left[ \sin \frac{\pi}{5} + \frac{1}{2} \left( \cotg \frac{2\pi}{5} - \frac{1}{\sin \frac{\pi}{5}} - 1 \right) \right] \left( \sin \frac{\pi}{5} - \sin \frac{2\pi}{5} \right) - \]

\[ \left( 1 - 2\cotg \frac{2\pi}{5} \right) \left( \cos \frac{2\pi}{5} + \cos \frac{\pi}{5} \right) + \frac{\pi}{5} \left( \frac{2}{\cos \frac{2\pi}{5}} - \frac{1}{\sin \frac{\pi}{5}} - \cotg \frac{2\pi}{5} \right) \right] + \]

\[ \frac{\pi m^2}{40} \left( \frac{4\pi}{5} + \sin \frac{2\pi}{5} - \sin \frac{\pi}{5} \right), \]  \tag{17} 

we obtain the probability that we want.

References


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