Von Neumann-Morgenstern Solutions for 1-Convex Games

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Abstract

The 1-convex TU games have appealing theoretical and practical applications. We identify the family of 1-convex n-person games with the same cores but different vNM solutions. A characterization of the set of imputations which are not dominated by the core and an explicit description of particular vNM solutions are given. These solutions consist of the core and complementary set which allows a simple interpretation.

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1 Introduction and definitions

The core [3] and vNM stable sets [4] are the fundamental solution concepts in cooperative game theory. But both are open to criticism. First of all, they may not exist. The core can consists of 'tyrannical' allocation that gives all of society’s resources to a single agent. There are no universal algorithm for computing the vNM solutions and a nonemptiness criterion. The complexity of vNM solutions is such that results only were obtained for special cases. As far as we know, vNM solutions of 1-convex games [1]-[2] not yet seen in the literature. In this paper we concentrate on the family of (0,1)-normalized 1-convex games. Our aim is to construct vNM solutions for some games of this family and provide their interpretation.
Let’s recall the facts that will be used later. A transferable utility game with player set \( N = \{1, \ldots, n\} \) is defined by a function \( \nu : 2^N \to \mathbb{R} \) satisfying \( \nu(\emptyset) = 0 \). The set of such games is denoted by \( \mathcal{G}^N \). For coalition \( K \in 2^N \setminus \{\emptyset\} \) denote its cardinality by \( |K| \). For \( x \in \mathbb{R}^N \) and \( K \in 2^N \setminus \{\emptyset\} \) let \( x(K) = \sum_{i \in K} x_i \). We will use \( x(i, \ldots, k), K \cup i \) to denote \( x(\{i, \ldots, k\}), K \cup \{i\} \), etc. Let \( m^\nu = (m_i^\nu)_{i \in N} \), where \( m_i^\nu = \nu(N) - \nu(N \setminus i) \) is a marginal contribution of player \( i \) to the grand coalition \( N \). A game \( \nu \in \mathcal{G}^N \) is:

- \( N \)-essential if \( \sum_{i \in N} \nu(i) < \nu(N) \);
- symmetric if \( \nu(K) = \nu(S) \) whenever \( |K| = |S| \);
- \((0,1)\)-normalized if \( \nu(N) = 1 \) and \( \nu(i) = 0, i \in N \);
- convex if \( \nu(K) + \nu(S) \leq \nu(K \cup S) + \nu(K \cap S) \), \( K,S \in 2^N \);
- 1-convex if \( m^\nu(N) \geq \nu(N) \), \( \nu(K) + m^\nu(N \setminus K) \leq \nu(N) \), \( K \in 2^N \setminus \{\emptyset\} \).

The imputation set, dual imputation set and core of a game \( \nu \) are \( I(\nu) = \{x \in X(\nu) : x_i \geq \nu(i), i \in N\} \), \( I^*(\nu) = \{x \in X(\nu) : x_i \leq m_i^\nu, i \in N\} \), \( C(\nu) = \{x \in X(\nu) : x(S) \geq \nu(S), S \in 2^N \setminus \{N, \emptyset\}\} \), where \( X(\nu) = \{x \in \mathbb{R}^N : x(N) = \nu(N)\} \). For \( x,y \in I(\nu) \), \( x \) dominates \( y \) via coalition \( K \) \((x \succ_K y)\), if \( x(K) \leq \nu(K) \) and \( x_i > y_i, i \in K \). An imputation \( x \) dominates \( y \) \((x \succ y)\), if there is a coalition \( K \) such that \( x \succ_K y \). For any \( A \subseteq I(\nu) \) denote by \( Dom_K A \) the set consisting of all imputations that are dominated via \( K \) by some element in \( A \) and let

\[ \Omega = \{K \in 2^N : 2 \leq |K| \leq n - 1\}. \]

Then \( DomA = \bigcup_{K \in \Omega} Dom_K A \). The vNM solution \( NM(\nu) \) of a game \( \nu \) is defined by two conditions

\[ NM(\nu) \cap Dom NM(\nu) = \emptyset \quad \text{(internal stability),} \]
\[ NM(\nu) \cup Dom NM(\nu) = I(\nu) \quad \text{(external stability).} \]

Let \( \mathcal{NM}(\nu) \) be a collection of vNM solutions of \( \nu \in \mathcal{G}^N \). A nonempty core consists of undominated imputations, i.e. it is internally stable. An externally stable core is called stable. It coincides with the (unique) stable set.

### 2 Results

A game \( \nu \in \mathcal{G}^N \) with \( \sum_{i \in N} \nu(i) = \nu(N) \) is out of interest. Any \( N \)-essential game is strategically equivalent to the unique game in \((0,1)\)-form and the core as well as vNM solutions are relative invariant w.r.t. strategic equivalence. It is then natural to focus attention on the polytope \( \mathcal{P}^N \subset \mathcal{G}^N \) of nonnegative \((0,1)\)-normalized games. Denote by \( \mathcal{CP}^N \) and \( 1\mathcal{CP}^N \) the subsets of convex and 1-convex games in \( \mathcal{P}^N \).

First we identify the family of nonsymmetric 1-convex games generated by symmetric convex one and show that their cores (except for convex game) are not stable.
Theorem 2.1. Let $n \geq 4$, $H \in \Omega$, $|H| \leq n - 2$,

$$\hat{m} = \frac{1}{n - 1}$$

and

$$\omega(K) = \begin{cases} (|K| - 1)\hat{m}, & K \in 2^N \setminus \{\emptyset\}, \\ 0, & K = \emptyset, \end{cases}$$

$$\nu_H(K) = \begin{cases} \omega(K), & K \not\subseteq H, \\ 0, & K \subseteq H. \end{cases}$$

Then:

(i) $1CP^N \cap CP^N = \{\omega\}$, $\mathcal{N}\mathcal{M}(\omega) = \{C(\omega)\}$;

(ii) $\nu_H \in 1CP^N$, $I(\nu_H) = I(\omega) = \{x \in \mathbb{R}_+^N : x(N) = 1\}$;

(iii) $C(\nu_H) = C(\omega) = \{x \in I(\omega) : x_i \leq \hat{m}, i \in N\}$;

Proof. (i) The first statement was proved in [5]. The second one follows by the convexity of $\omega$.

(ii) Obviously $\omega, \nu_H \in \mathcal{P}^N$. Therefore, $I(\nu_H) = I(\omega)$ and these sets are of above form. Since $m^{\nu_H}(N) = \frac{n}{n-1} > 1$ and for all $K \in 2^N \setminus \{\emptyset\}$, $\nu_H(K) + m^{\nu_H}(N \setminus K) \leq \omega(K) + (n - |K|)\hat{m} = 1$ it follows that $\nu_H \in 1CP^N$. The core of any 1-convex game coincides with the dual imputation set [2]. The equalities $m^{\nu_H}_i = \hat{m}, i \in N$, imply $C(\nu_H) = C(\omega)$ and the core's representation.

(iii) Let

$$\Omega^1_H = \{K \in \Omega : K \subseteq H\}, \quad \Omega^2_H = \{K \in \Omega : K \not\subseteq H\}.$$ 

Define $y \in I(\nu_H) \setminus C(\nu_H)$ by

$$y_i = \begin{cases} \frac{1}{n-|H|}, & i \in N \setminus H, \\ 0, & i \in H. \end{cases}$$

Using $\nu_H(K) = 0$, $K \in \Omega^1_H$, we see that $y \notin \bigcup_{K \in \Omega^1_H} \text{Dom}_K C(\nu_H)$. If $K \in \Omega^2_H$, then $y_i > \hat{m}$ for some $i \in K$. Hence, $y \notin \bigcup_{K \in \Omega^2_H} \text{Dom}_K C(\nu_H)$. Finally, $y \notin \text{Dom} C(\nu_H)$. \hfill $\Box$

In order to describe the vNM solution for game $\nu_H$ we need to characterize the subset of imputations of game $\omega$ which are dominated by $C(\omega)$ via fixed coalition $K \in \Omega$ (Lemma 2.2) and the subset of imputations of game $\nu_H$ which are undominated by $C(\nu_H)$ (Lemma 2.3).

Lemma 2.2. Let $K \in \Omega$. Then $x \in \text{Dom}_K C(\omega)$ iff $x \in I(\omega)$ and the following condition holds

$$x(K) < (|K| - 1)\hat{m}; \quad x_i < \hat{m}, \quad i \in K. \tag{1}$$
Proof. Take $x \in I(\omega)$ satisfying (1). Then $\Delta = (|K| - 1)\bar{m} - x(K) > 0$ and $b_i = \bar{m} - x_i > 0$, $i \in K$. Consider $y \in \mathbb{R}^N$, where

$$y_i = \begin{cases} x_i + \epsilon_i, & i \in K, \\ \bar{m}, & i \in N \setminus K, \end{cases}$$

and

$$\sum_{i \in K} \epsilon_i = \Delta, \; 0 < \epsilon_i \leq b_i, \; i \in K.$$

Since $\sum_{i \in K} b_i > \Delta$, the above system is compatible. Obviously $y \in I(\omega)$. Using Theorem 2.1(ii) and $y_i \leq \bar{m}$, $i \in N$, we obtain $y \in C(\omega)$. Moreover, $y \succ_{K} x$ because $y(K) = (|K| - 1)\bar{m} = \omega(K)$. Consequently, $x \in \text{Dom}_K C(\omega)$. Conversely, let $x \in \text{Dom}_K C(\omega)$. Then $x \in I(\omega)$ and there exists $y \in C(\omega)$ such that $y \succ_{K} x$, i.e. $x_i < y_i \leq \bar{m}$, $i \in K$, and $x(K) < y(K) = \omega(K)$. Hence, $x$ satisfies (1). \qed

Lemma 2.3. Let $n \geq 4$, $H \in \Omega$, $|H| \leq n - 2$,

$$L = (I(\nu_H) \setminus C(\nu_H)) \setminus \text{Dom}_C(\nu_H).$$

Then $L = \bigcup_{K \in \Omega_H^1} L_K$, where $L_K$ consists of all $x \in I(\nu_H)$ satisfying (1) and

$$x_i \geq \bar{m}, \; i \in N \setminus H; \quad (2)$$

$$\max_{i \in N \setminus K} x_i > \bar{m}. \quad (3)$$

Proof. By Theorem 2.1(iii), $L \neq \emptyset$. By Theorem 2.1(ii) and $\nu_H(K) = \omega(K)$ for $K \in \Omega_H^2$, we have $\bigcup_{K \in \Omega_H^2} \text{Dom}_K C(\nu_H) = \bigcup_{K \in \Omega_H^2} \text{Dom}_K C(\omega)$. Since $I(\nu_H) \subset \mathbb{R}_+^N$ and $\nu_H(K) = 0$ for $K \in \Omega_H^1$, we have $\bigcup_{K \in \Omega_H^1} \text{Dom}_K C(\nu_H) = \emptyset$. So

$$\text{Dom}_C(\nu_H) = \bigcup_{K \in \Omega_H^1} \text{Dom}_K C(\omega).$$

We know that

$$\text{Dom}_C(\omega) = I(\omega) \setminus C(\omega) = (\bigcup_{K \in \Omega_H^1} \text{Dom}_K C(\omega)) \cup (\bigcup_{K \in \Omega_H^2} \text{Dom}_K C(\omega)).$$

Consequently,

$$L = (I(\omega) \setminus C(\omega)) \setminus (\bigcup_{K \in \Omega_H^1} \text{Dom}_K C(\omega))$$

$$= (\bigcup_{K \in \Omega_H^1} \text{Dom}_K C(\omega)) \setminus (\bigcup_{K \in \Omega_H^2} \text{Dom}_K C(\omega)). \quad (4)$$

Take $x \in L$. Then $x \in I(\nu_H) = I(\omega)$ and there exists $K \in \Omega_H^1$ such that $x \in \text{Dom}_K C(\omega)$. By Lemma 2.2, $x$ satisfies (1). Since $x \notin C(\nu_H)$ then, by (1) and Theorem 2.1(ii), there must exists at least one $l \in N \setminus K$ such that $x_l > \bar{m}$. So $x$ satisfies (3). If there is $r \in (N \setminus H) \setminus l$ such that $x_r < \bar{m}$, then using (1) we see that $x(K \cup r) < |K|\bar{m}$. By Lemma 2.2, $x \in \text{Dom}_{K \cup r} C(\omega)$. This contradicts (4) because $K \cup r \in \Omega_H^2$. So $x$ satisfies (2). We proved that $x \in L_K$, i.e. the inclusion $L \subseteq \bigcup_{K \in \Omega_H^1} L_K$ holds. Conversely, suppose $x \in L_K$ for some $K \in \Omega_H^1$. Then $x \in I(\nu_H)$ and (1)-(3) hold. By Theorem 2.1(ii) and (3), $x \notin C(\nu_H)$. Any coalition $S \in \Omega_H^2$ contains at least one element of $N \setminus H$. By (2) and Lemma 2.2, $x \notin \bigcup_{S \in \Omega_H^2} \text{Dom}_S C(\omega)$. Using (4) we obtain $x \in L$. Hence, $L \supseteq \bigcup_{K \in \Omega_H^1} L_K$. \qed
We see that (5) because $y$ such the contrary, that remains to prove that $\Omega$. We can write $\nu_{NM}$ solution for game $\nu$.

**Theorem 2.4.** Let $n \geq 4$, $H \in 2^N$, $|H| = 2$. Then

$$Q_H = C(\nu_H) \cup F_H \in N.M(\nu_H),$$

where the set $F_H$ consists of all $x \in I(\nu_H)$ satisfying

$$x(H) < (|H| - 1)\bar{m},$$

(5)

$$x_i = \mu^x = \frac{1 - x(H)}{n - |H|}, \quad i \in N \setminus H.$$  

(6)

**Proof.** As will become clear below $F_H \neq \emptyset$. Take $x \in F_H$. From $|H| = 2$, (5) and $x \in \mathbb{R}^N_+$ it follows that $x_i < \bar{m}$, $i \in H$. Using (5) and (6) we obtain $\mu^x > \frac{1 - \bar{m}}{n - 2} = \bar{m}$. So $x$ satisfies (2) and conditions (1),(3) with $K = H$. This implies $x \in L_H$. Thus $F_H \subseteq L_H$. Clearly, $F_H \neq L_H$. Besides, $L_H = L$ because $\Omega^1_H = \{H\}$ for coalition $H$ with $|H| = 2$.

To prove internal stability of $Q_H$, note that, by Lemma 2.3, $DomC(\nu_H) \cap L = \emptyset$. Consequently, $Dom(\nu_H) \cap F_H = \emptyset$. Since $Dom(\nu_H) \cap C(\nu_H) = \emptyset$, it remains to prove that $F_H \cap DomF_H = \emptyset$. Take $x, y \in F_H$, $x \neq y$. Suppose, to the contrary, that $x \succ y$ for some $S \in \Omega$. We can write $S = S' \cup S''$, where $S' \subseteq N \setminus H$, $S'' \subseteq H$. Then:

- $S' \neq \emptyset$ because $y \in \mathbb{R}^N_+$ and $\nu_H(K) = 0$, $K \subseteq H$;
- $S'' \neq \emptyset$ because otherwise $x(S) = x(S') = |S'|\mu^x > |S'|\bar{m} > \nu_H(S)$, which contradicts $x(S) \leq \nu_H(S)$;
- if $|S''| = 1$ we have $x(S) = |S'|\mu^x + x(S'') > |S'|\bar{m} = \nu_H(S)$ a contradiction;
- if $S'' = H$, then $x(H) = 1 - (n - 2)\mu^x < 1 - (n - 2)\bar{m} = y(H)$, which contradicts $x_i > y_i$, $i \in H$.

To prove external stability of $Q_H$ it suffices (in view of Lemma 2.3) to show that $L \setminus F_H \subseteq DomF_H$. Let $x_{\min} = \min_{i \in N \setminus H} x_i$ and $H = \{1, 2\}$. Take $x \in L \setminus F_H = L_H \setminus F_H$. Then $x$ belongs to $I(\nu_H)$ and satisfies (1) with $K = H$, i.e. $x$ satisfies (5). But $x$ does not satisfy (6) since otherwise $x \in F_H$ holds, which gives a contradiction. Consequently, $\Delta = x(N \setminus H) - (n - 2)x_{\min} > 0$. Define $y \in \mathbb{R}^N$, where

$$y_i = \begin{cases} 
\mu^y = x_{\min} + \beta, & i \in N \setminus H, \\
x_i + \alpha_i, & i \in H,
\end{cases}$$

(7)

$$\alpha_1, \alpha_2 > 0, \quad \alpha_1 + \alpha_2 < \min\{\Delta, \bar{m} - x(H)\}, \quad \beta = \frac{\Delta - \alpha_1 - \alpha_2}{n - 2} > 0.$$ 

Such $\alpha_1, \alpha_2$ obviously exist, $y \in I(\nu_H)$ and (6) holds. The vector $y$ satisfies (5) because $y(H) = x(H) + \alpha_1 + \alpha_2 < \bar{m}$. Thus $y \in F_H$. Let $i_{\min} \in \arg \min_{i \in N \setminus H} x_i$.

We see that $y \succ y_{i_{\min}}$ in the set because $y(H \cup i_{\min}) = \frac{1 + (n - 3)(x(H) + \alpha_1 + \alpha_2)}{n - 2} < \frac{1 + (n - 3)\bar{m}}{n - 2} = 2\bar{m} = \nu_H(H \cup i_{\min}).$
Our first example demonstrates that Theorem 2.4 cannot be extended to game \( \nu_H \) with \(|H| > 2\).

**Example 2.5.** Let \( N = \{1, \ldots, 5\} \) and \( H = \{1, 2, 3\} \). Then \( \hat{m} = \frac{1}{4}, \nu_H(K) = \frac{|K| - 1}{4} \) whenever \( K \not\subseteq H \), \( \nu_H(K) = 0 \) otherwise. The imputations

\[
x = \left( \frac{1}{8}, 0, 0, \frac{7}{16}, \frac{7}{16} \right), \quad y = \left( \frac{1}{64}, \frac{1}{64}, \frac{29}{64}, \frac{29}{64} \right)
\]

satisfy (5)-(6) because \( x(H) = \frac{1}{8} < 2\hat{m} = \frac{1}{2}, y(H) = \frac{3}{32} < \frac{1}{2}, x_4 = x_5, y_4 = y_5 \). So \( x, y \in F_H \). However, \( y \succ_{\{2,3,4\}} x \) because \( y(2, 3, 4) = \frac{37}{64} < \nu_H(2, 3, 4) = \frac{1}{2} \) and \( y_i > x_i, i \in \{2, 3, 4\} \). Thus, \( F_H \) violates the internal stability property.

The last theorem describes vNBM solution for five-person game \( \nu_H \) with \(|H| = 3\).

**Theorem 2.6.** Let \( n = 5, H \in 2^N, |H| = 3 \). Then

\[
W_H = C(\nu_H) \cup P_H \in \mathcal{N}\mathcal{M}(\nu_H),
\]

where the set \( P_H \) consists of all \( x \in I(\nu_H) \) satisfying (5)-(6) and

\[
\mu^x + x(S) \geq 2\hat{m}, \text{ } S \subset H, \text{ } |S| = 2.
\]

**Proof.** Some statements that will be used in the following proof are similar to those in proof of Theorem 2.4. They are given without the explanations. Take \( x \in P_H \not\in \emptyset \). Let \( H = \{1, 2, 3\} \). Suppose \( x_i \geq \hat{m} \) for some \( i \in H \). For example, \( x_1 \geq \hat{m} \). By (5) and (6), \( \mu^x > \frac{1 - 2\hat{m}}{2} = \hat{m} \). By (8), \( x_2 + x_3 \geq 2\hat{m} - \frac{1 - x(H)}{2} = \frac{x(H)}{2} \).

This implies that \( x_2 + x_3 \geq x_1 \geq \hat{m} \) and \( x(H) \geq 2\hat{m} = (|H| - 1)\hat{m} \), which contradicts (5). Thus \( x_i < \hat{m}, i \in H \). We have \( P_H \subset L_H \subset L \).

To prove internal stability of \( W_H \) we need only to show that \( P_H \cap DomP_H = \emptyset \). Take \( x, y \in P_H, x \not= y \). Suppose, to the contrary, that \( x \succ y \) for some \( S \in \Omega \). Then \( S = S' \cup S'' \), where \( \emptyset \not\subseteq S' \subseteq N \setminus H, \emptyset \not\subseteq S'' \subseteq H \). If \( |S''| = 1 \) and \( S'' = H \), then \( y \) is not dominated by \( x \). If \( |S''| = 2 \), then using (8) and \( \mu^y > \hat{m} \) we obtain \( x(S) > y(S) = |S'|\mu^y + y(S'') \geq |S'|\mu^y + 2\hat{m} - \mu^y = (|S'| - 1)\mu^y + 2\hat{m} > (|S'| + 1)\hat{m} = x_H(S), \) which gives a contradiction.

To prove external stability of \( W_H \) it suffices to show that \( L \setminus P_H \subset DomP_H \).

Take now \( x \in L \setminus P_H \subset I(\nu_H) \). We distinguish the following cases.

**Case 1:** \( x \in L_H \), i.e. \( x(H) < 2\hat{m}, x_{\min} = \min_{i \in N \setminus H} x_i \geq \hat{m}, \text{ max } x_i > \hat{m} \) and \( x_i < \hat{m}, i \in H \). In this case, \( x \) satisfies (5).

**Subcase 1:** there exists \( S \subset H \) with \( |S| = 2 \) such that \( x_{\min} + x(S) < 2\hat{m} \).

Let \( S = \{2, 3\} \). Consider \( y \in \mathbb{R}^N \), where

\[
y_i = \begin{cases} \mu^y = x_{\min} + \beta, & i \in \{4, 5\}, \\ \alpha_i, & i = 1, \\ x_i + \alpha_i, & i \in \{2, 3\}, \end{cases}
\]
\[
\begin{align*}
\alpha_2 + \alpha_3 &< 2\bar{m} - x_{\min} - x(S), & \alpha_i > 0, & i \in S, \\
\alpha_1 &= x(S) + \alpha_2 + \alpha_3, & \beta &= 2\bar{m} - x_{\min} - \alpha_1 > 0.
\end{align*}
\] (10)

Under the above assumptions, such \(\alpha_2, \alpha_3\) exist, \(0 < \alpha_1 = 2\bar{m} - x_{\min} - \beta \leq \bar{m} - \beta < \bar{m}\) and \(y(H) = 2\alpha_1 < 2\bar{m}\). We see that \(y \in I(\nu_H)\) and (5)-(6) hold. Further, \(\mu^y + y(S) = x_{\min} + \beta + \alpha_1 = 2\bar{m}\) and \(\mu^y + y(1, 2) = x_{\min} + \beta + \alpha_1 + x_2 + \alpha_2 > 2\bar{m}\). Similarly \(\mu^y + y(1, 3) > 2\bar{m}\). Hence \(y\) satisfies (8). We obtain \(y \in P_H\). Moreover, \(y \succ_{\{2,3,4\}} x\).

**Subcase 2**: \(x_{\min} + x(S) \geq 2\bar{m}\) for all \(S \subset H\) with \(|S| = 2\). Then \(x_4 \neq x_5\) since otherwise \(x \in P_H\). Define \(y\) by formula (7), where

\[
2\beta + \sum_{i \in H} \alpha_i = x(N \setminus H) - 2x_{\min}, \quad \beta > 0, \quad \alpha_i > 0, \quad i \in H.
\]

The last system is solvable because \(x_4 \neq x_5\) implies \(x(N \setminus H) - 2x_{\min} > 0\). Clearly, \(y \in I(\nu_H)\). Further, \(y(H) = x(H) + \sum_{i \in H} \alpha_i = 1 - 2(\beta + x_{\min}) < 1 - 2x_{\min} \leq 2\bar{m}\). Hence, (5)-(6) hold. Since \(\mu^y + y(S) > x_{\min} + x(S) \geq 2\bar{m}\) for all \(S \subset H\) with \(|S| = 2\), then \(y\) satisfies (8). We obtain \(y \in P_H\). Let \(i_{\min} \in \arg \min_{i \in N \setminus H} x_i\). Using \(y(H) \cup i_{\min}) = x_{\min} + \beta + y(H) = \frac{1 + y(H)}{2} < 1 + \frac{2\bar{m}}{2} = 3\bar{m} = \nu_H(H \cup i_{\min})\), we see that \(y \succ_{H \cup i_{\min}} x\).

**Case 2**: \(x \notin L_H\). Then \(x \in L_K\) for some coalition \(K \subset H\) with \(|K| = 2\). By Lemma 2.3 \(x(K) < \bar{m}\), \(x_i < \bar{m}\) for all \(i \in K\), \(x_i > \bar{m}\) for all \(i \in N \setminus H\) and \(i \in N \setminus H\) with \(m\). Let \(K = \{2, 3\}\). If \(x_1 < \bar{m}\) we obtain \(x \in L_H\) a contradiction. Consequently, \(x_1 \geq \bar{m}\). Define \(y\) by formula (9), where \(\alpha_i, i \in H\), and \(\beta\) are determined by (10) with \(S = K\). Such \(\alpha_2, \alpha_3\) exist because \(2\bar{m} - x_{\min} - x(K) \geq 2\bar{m} - \frac{1 - x(H)}{2} - x(K) = \frac{x_{1} - x(K)}{2} > 0\). As in subcase 1 of this theorem, we can prove that \(y \in P_H\) and \(y \succ_{\{2,3,4\}} x\).

The second example shows that Theorem 2.6 cannot be extended to game \(\nu_H\) with \(n \geq 6\) and \(|H| = 3\).

**Example 2.7.** Let \(N = \{1, ..., 6\}\) and \(H = \{1, 2, 3\}\). Then \(\bar{m} = \frac{1}{5}\), \(\nu_H(K) = \frac{|K| - 1}{5}\) whenever \(K \subseteq H\), \(\nu_H(K) = 0\) else. The imputation

\[
y = (0, 0, \frac{1}{10}, \frac{3}{10}, \frac{3}{10}, \frac{3}{10})
\]

does not belong to the core because \(\mu^y = \frac{3}{10} \not\succ \bar{m}\). Since \(\mu^y + y(1, 2) = \frac{3}{10} < 2\bar{m}\), the condition (8) does not hold, i.e. \(y \notin P_H\). So \(y \notin W_H\). As in the proof of Theorem 2.6 we see that dominations can be done only via coalitions \(\{1, 2, j\}\), \(j \in N \setminus H\). Pick \(\{1, 2, 4\}\) and assume the existence of imputation \(x \in W_H\) such that \(x \succ_{\{1,2,4\}} y\). Then \(x(1,2,4) \leq \nu_H(1,2,4) = 2\bar{m}\). Since \(x_4 > y_4 > \bar{m}\) we obtain \(x \notin C(\nu_H)\). Hence, \(x \in P_H\). From (8) and \(x(1,2,4) \leq 2\bar{m}\) it follows that \(x(1,2,4) = 2\bar{m}\). By (5)-(6), \(x_5 = x_6 = \mu^x > \bar{m}\). Consequently,
\[ x(N) = 2\dot{m} + 2\mu x + x > 4\dot{m} = 1, \text{ which contradicts } x \in I(\nu_H). \] Thus the set \( P_H \), violates the external stability property.

## 3 Results interpretation and conclusion

The construction and interpretation of vNM solutions are important from the viewpoint of economic applications and theory. Any game \( \nu_H \) is symmetric w.r.t. the coalitions \( H \) and \( N \setminus H \), but it does not symmetric as a whole. The average marginal contribution of a participant of coalition \( H \) (small agent) is less than that of a participant of \( N \setminus H \) (big agent). The Shapley value \( Sh(\nu_H) \) reflects this fact: \( Sh_i(\nu_H) < Sh_j(\nu_H) \) for all \( i \in H, j \in N \setminus H \). All 1-convex games considered here have the identical cores that are symmetric w.r.t. the grand coalition and coincide with the core of convex game \( \omega \). The cores contain special allocations where the payoffs of all small agents are the maximum they could get (i.e. equal to \( \dot{m} \)), while one of big agents gets nothing. Unlike the core, every imputation in \( F_H \) or \( P_H \) assigns to a big agent more than \( \dot{m} \) while the payoff of each small agent is less than \( \dot{m} \). So, they prescribe a rather natural and intuitive outcomes.

Note that for \( \nu_H \) with \( n = 5 \) and \( |H| = 3 \), every \( x \in I(\nu_H) \) satisfying (6),(8) and \( x(H) = (|H| - 1)\dot{m} \) belongs to \( C(\nu_H) \). Similarly, for \( \nu_H \) with \( |H| = 2 \), every \( x \in I(\nu_H) \) satisfying (6) and the last equality, belongs to \( C(\nu_H) \). Thus, in spite of strict inequality (5), \( Q_H \) and \( W_H \) are the closed sets in \( \mathbb{R}^N \).

Let's look at the game \( \nu_H \) with \( n = 5 \) and \( |H| = 3 \). Assume that the grand coalition forms and players agree on using the vNM solution as stability concept. The following decision-making process is possible.

**First stage.** The players decide, must the outcome belong to the core or not. If yes, they chose a core selector and the game ends. Otherwise, the players partition themselves into subcoalitions: the union \( H \) of small agents and union \( N \setminus H \) of big ones.

**Second stage.** The big agents bargain about the portion \( x(H) < (|H| - 1)\dot{m} \) of cooperation surplus that must get the union \( H \). No player in \( H \) can participate in this stage of negotiations. The big agents union’s power is such that \( x(H) \) can be equal to zero. When union \( N \setminus H \) has made his decision, the value \( \nu_H(N) - x(H) \) is divided equally among its members, i.e. each of them receives \( \mu x > \dot{m} \). After that all big agents leave the game.

**Last stage.** The small agents discuss how to share \( x(H) \). They play the reduced game \((H,\nu_H)\), where

\[
\nu_H^*(K) = \begin{cases} 
  x(H), & K = H, \\
  2\dot{m} - \mu x, & |K| = 2, \\
  0, & \text{else.}
\end{cases}
\]
Note that $\nu_H$ satisfies the necessary and sufficient nonemptiness condition

$$\frac{\nu_H(K)}{|K|} \leq \frac{x(H)}{|H|}, \; \emptyset \neq K \subset H,$$

for the core of symmetric game. Indeed, this is true for a singleton coalition $K$. If $|K| = 2$ we have $\frac{2m - \mu'}{2} \leq \frac{1 - 2\mu'}{3}$ or $\mu' \leq 2\tilde{m}$. The last inequality holds because $\mu' = \frac{1 - x(H)}{2} = 2\tilde{m} - \frac{x(H)}{2}$ and $x(H) \geq 0$. The small agents chose the final bargaining outcome in the core of reduced game.

A number of questions left open.

- Have the games considered in Theorem 2.4 and Theorem 2.6 the unique vNM solution? If no, that how other vNM solutions look like?
- What are the vNM solutions of game $\nu_H$ with $n \geq 6$ and $|H| \geq 3$? Lemma 2.3 can be helpful for constructing such solutions.

It is of interest to find the explicit form of vNM solutions for games with player set $N$, integer characteristic functions $\bar{\omega} = \omega/\tilde{m}$, $\bar{\nu}_H = \nu_H/\tilde{m}$ and integer side payments (discrete games). Even the convexity of $\bar{\omega}$ is not sufficient for stability of the core [6]. However, we know that any discrete game has a finite number of vNM solutions and they can be calculated with the graph theory algorithms.

References


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