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Functional Form of the Jensen and the Hermite-Hadamard Inequality

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Abstract

The paper mainly deals with convex functions of several variables. Some variants of the Jensen type inequalities and the generalization of the Hermite-Hadamard inequality are obtained. The work is based on the adapted and extended McShane's functional form of Jensen's inequality and its interesting applications.

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1. Introduction

Let $\mathcal{X}$ be a non-empty set. Let $\mathbb{X}$ be a real vector space of functions $g : \mathcal{X} \to \mathbb{R}$. We consider the space $\mathbb{X}$ containing the unit function e_0 defined with $e_0(x) = 1$ for every $x \in \mathcal{X}$.

A linear functional $L : \mathbb{X} \to \mathbb{R}$ is said to be positive (non-negative) or monotone if $L(g) \geq 0$ for every non-negative function $g \in \mathbb{X}$. We use the positive functional L satisfying $L(e_0) = 1$. Such functional is called unital or normalized.

In 1931, Jessen formulated in [4] the functional form of Jensen's inequality for convex functions of one variable as follows.

Theorem A. *Let $\mathcal{I} \subseteq \mathbb{R}$ be a closed interval. Let $g \in \mathbb{X}$ be a function such that $g(x) \in \mathcal{I}$ for every $x \in \mathcal{X}$. Let $f : \mathcal{I} \to \mathbb{R}$ be a continuous convex function such that $f(g) \in \mathbb{X}$.*

Then every positive unital linear functional $L : \mathbb{X} \to \mathbb{R}$ satisfies the inclusion

$$L(g) \in \mathcal{I}, \tag{1.1}$$

and the inequality

$$f(L(g)) \leq L(f(g)). \tag{1.2}$$

If the function f is concave, then the reverse inequality is valid in (1.2).

The inclusion in (1.1) is generally not true if the interval $\mathcal{I}$ is not closed. The inequality in (1.2) can be proved using support lines of the function f at interior points of the interval $\mathcal{I}$. The inequality in (1.2) is generally not valid if the function f is not continuous.

In 1937, McShane extended in [5, Theorems 1-2] the functional form of Jensen's inequality to convex functions of several variables. He has covered the generalization in two steps, calling them the geometric (the inclusion in (1.3)) and analytic (the inequality in (1.4)) formulation of Jensen's inequality as follows.

Theorem B. *Let $\mathcal{C} \subseteq \mathbb{R}^m$ be a closed convex set. Let $g_j \in \mathbb{X}$ be functions such that $(g_1(x), \dots, g_m(x)) \in \mathcal{C}$ for every $x \in \mathcal{X}$. Let $f : \mathcal{C} \to \mathbb{R}$ be a continuous convex function such that $f(g_1, \dots, g_m) \in \mathbb{X}$.*

Then every positive unital linear functional $L : \mathbb{X} \to \mathbb{R}$ satisfies the inclusion

$$(L(g_1), \dots, L(g_m)) \in \mathcal{C}, \tag{1.3}$$

and the inequality

$$f(L(g_1), \dots, L(g_m)) \leq L(f(g_1, \dots, g_m)). \tag{1.4}$$

If the function f is concave, then the reverse inequality is valid in (1.4).

The hyperplanes that contain the set $\mathcal{C}$ were used in the proof of the inclusion in (1.3). The epigraph of the function f,

$$\mathrm{epi}(f) = \{(x_1, \dots, x_m, x_{m+1}) \in \mathcal{C} \times \mathbb{R} \,|\, x_{m+1} \geq f(x_1, \dots, x_m)\}, \tag{1.5}$$

and the inclusion in (1.3) were applied in the proof of the inequality in (1.4).

2. Overview of Results Using Jessen's Functional Form

Let $[a,b] \subset \mathbb{R}$ be a bounded closed interval where $a < b$. In this section, we use the space $\mathbb{X} = C([a,b])$ of all continuous functions $g : [a,b] \to \mathbb{R}$. The identity function in $C([a,b])$ will be denoted by e_1, so $e_1(x) = x$ for every $x \in [a,b]$.

Every number $x \in \mathbb{R}$ can be uniquely presented as the binomial affine combination

$$x = \frac{b-x}{b-a}a + \frac{x-a}{b-a}b \tag{2.1}$$

which is convex if, and only if, the number x belongs to the interval $[a,b]$. Given the function $f : \mathbb{R} \to \mathbb{R}$, let $f^{\text{line}}_{\{a,b\}} : \mathbb{R} \to \mathbb{R}$ be the function of the line passing through the points $A(a, f(a))$ and $B(b, f(b))$ of the graph of f. Using the affinity of $f^{\text{line}}_{\{a,b\}}$, we get

$$f^{\text{line}}_{\{a,b\}}(x) = \frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b). \tag{2.2}$$

If the function f is convex, then we have the inequality

$$f(x) \le f^{\text{line}}_{\{a,b\}}(x) \ \text{ if } x \in [a,b], \tag{2.3}$$

and the reverse inequality if $x \notin (a,b)$. In what follows, we use convex combinations $\sum_{i=1}^n p_i x_i$ from $[a,b]$, that is to say all the points $x_i \in [a,b]$ and the non-negative coefficient sum $\sum_{i=1}^n p_i = 1$.

Considering the inequality in (2.3) we have the following adapted and extended version of Theorem A:

Theorem 2.1. *Let $g : [a,b] \to [a,b]$ be a continuous function, and $f : [a,b] \to \mathbb{R}$ be a continuous convex function.*

Then every positive unital linear functional $L : C([a,b]) \to \mathbb{R}$ satisfies the inequality

$$f(L(g)) \le L(f(g)) \le f^{line}_{\{a,b\}}(L(g)). \tag{2.4}$$

Proof. The right-hand side of the inequality in (2.4) follows from the inequality $f \le f^{\text{line}}_{\{a,b\}}$ of (2.3), and the affinity of the function $f^{\text{line}}_{\{a,b\}}$. □

The inequality in (2.4) can also be used in the prolonged form

$$f\left(\frac{b-L(g)}{b-a}a + \frac{L(g)-a}{b-a}b\right) \le L(f) \le \frac{b-L(g)}{b-a}f(a) + \frac{L(g)-a}{b-a}f(b). \tag{2.5}$$

Putting $g = e_1$ in the inequality in (2.4), we get the special inequality

$$f(L(e_1)) \le L(f) \le f^{\text{line}}_{\{a,b\}}(L(e_1)) \tag{2.6}$$

that can be taken as the functional form of the Jensen and chord inequality, and the Hermite-Hadamard inequality. This inequality has numerous applications to discrete and integral case. Some of them will be listed.

Jensen's and chord's inequality:

Corollary 2.2. *If $\sum_{i=1}^{n} p_i x_i$ is a convex combination from $[a,b]$, then every continuous convex function $f:[a,b]\to\mathbb{R}$ satisfies the inequality*

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \le \sum_{i=1}^{n} p_i f(x_i) \le f_{\{a,b\}}^{line}\left(\sum_{i=1}^{n} p_i x_i\right). \tag{2.7}$$

Convex function behaviour at the interval edges:

Corollary 2.3. *If convex combinations $\sum_{i=1}^{n} p_i x_i$ and $\alpha a+\beta b$ from $[a,b]$ have the same center*

$$\sum_{i=1}^{n} p_i x_i = \alpha a + \beta b, \tag{2.8}$$

then every continuous convex function $f[a,b]:\to\mathbb{R}$ satisfies the inequality

$$f(\alpha a+\beta b) \le \sum_{i=1}^{n} p_i f(x_i) \le \alpha f(a) + \beta f(b). \tag{2.9}$$

Hermite-Hadamard's inequality as a consequence of the convex function behavior at the edges:

Corollary 2.4. *Every continuous convex function $f:[a,b]\to\mathbb{R}$ satisfies the inequality*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}. \tag{2.10}$$

Corollaries 2.2-2.3 can be proved by using the summarizing positive unital linear functional

$$L^{\mathrm{sum}}(f) = L^{\mathrm{sum}}_{p_1x_1+\ldots+p_nx_n}(f) = \sum_{i=1}^{n} p_i f(x_i), \tag{2.11}$$

and Corollary 2.4 by using the integrating functional

$$L^{\mathrm{int}}(f) = L^{\mathrm{int}}_{[a,b]}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{2.12}$$

Using the integral method with convex combinations the discrete functional in (2.11) passes into the integral functional in (2.12). Applying the same method the Hermite-Hadamard inequality in (2.10) can be derived from the inequality in (2.9).

To generalize Corollaries 2.2-2.4 we take two functions, a positive continuous function $p:[a,b]\to\mathbb{R}$, and a continuous function $g:[a,b]\to[a,b]$. Then the generalization follows by using the summarizing or integrating functional

$$L_{p,g}(f) = \frac{L(pf(g))}{L(p)}. \tag{2.13}$$

Note that $L_{e_0,e_1} = L$ in both cases.

A general approach to means and their inequalities can be found in [1]. Discrete and integral forms of Jensen's type inequalities for convex functions of one variable have been studied in [10]. Some new Jensen's type inequalities have been offered in [9]. More on the Hermite-Hadamard inequality is written in [8] and [2], and a correct historic story on this inequality can be read in [7].

The generalized Jensen functional for the given functional $L : C([a,b]) \to \mathbb{R}$ is defined with

$$J_L(f) = L(f) - f(L(e_1)). \tag{2.14}$$

Applying the inequality in (2.6), we have the following global estimation (depending on the interval $[a,b]$ and the function f) of the bounds of J_L:

Corollary 2.5. *If $f : [a,b] \to \mathbb{R}$ is a continuous convex function, then every positive unital linear functional $L : C([a,b]) \to \mathbb{R}$ satisfies the inequality*

$$\begin{aligned} 0 \le J_L(f) &\le \left(f^{line}_{\{a,b\}} - f\right)(L(e_1)) \\ &\le \max_{x\in[a,b]} \left(f^{line}_{\{a,b\}} - f\right)(x). \end{aligned} \tag{2.15}$$

Global bounds for the discrete Jensen's functional have been investigated in [12], for discrete and integral functional in [3], and for generalized functional in [11]. The bounds in converses of Jensen's operator inequality have been considered and calculated in [6].

3. Main Results Using McShane's Functional Form

We assume $\mathbb{R}^m$ is the real vector space treating its points as the vectors with the standard coordinate addition $(x_1, \ldots, x_m) + (y_1, \ldots, y_m) = (x_1 + y_1, \ldots, x_m + y_m)$, and the scalar multiplication $\alpha(x_1, \ldots, x_m) = (\alpha x_1, \ldots, \alpha x_m)$.

If $A_1, \ldots, A_{m+1} \in \mathbb{R}^m$ are points such that the points $A_1 - A_{m+1}, \ldots, A_m - A_{m+1}$ are linearly independent, then the convex hull

$$\triangle = \operatorname{conv}\{A_1, \ldots, A_{m+1}\} \tag{3.1}$$

is called the m-simplex with the vertices $A_1, \ldots, A_{m+1}$. All the simplex vertices can not belong to the same hyperplane in $\mathbb{R}^m$. If we take the vertex coordinates $A_j(x_{j1}, \ldots, x_{jm})$, then any point $P(x_1, \ldots, x_m) \in \mathbb{R}^m$ can be presented by the unique affine combination

$$P = \sum_{j=1}^{m+1} \alpha_j A_j, \tag{3.2}$$

where the coefficients

$$\alpha_j = (-1)^{j+1} \frac{\begin{vmatrix} x_1 & \dots & x_m & 1 \\ x_{11} & \dots & x_{1m} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ x_{j-1\,1} & \dots & x_{j-1\,m} & 1 \\ x_{j+1\,1} & \dots & x_{j+1\,m} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ x_{m+1\,1} & \dots & x_{m+1\,m} & 1 \end{vmatrix}}{\begin{vmatrix} x_{11} & \dots & x_{1m} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ x_{j-1\,1} & \dots & x_{j-1\,m} & 1 \\ x_{j1} & \dots & x_{jm} & 1 \\ x_{j+1\,1} & \dots & x_{j+1\,m} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ x_{m+1\,1} & \dots & x_{m+1\,m} & 1 \end{vmatrix}}. \tag{3.3}$$

The combination in (3.2) is convex if, and only if, the point P belongs to the m-simplex $\triangle$.

Remark 3.1. *The coefficient formula in* (3.3) *can be reached by generalization the coefficients*

$$\alpha = \frac{\begin{vmatrix} x & 1 \\ b & 1 \end{vmatrix}}{\begin{vmatrix} a & 1 \\ b & 1 \end{vmatrix}}, \ \beta = -\frac{\begin{vmatrix} x & 1 \\ a & 1 \end{vmatrix}}{\begin{vmatrix} a & 1 \\ b & 1 \end{vmatrix}} \tag{3.4}$$

of the binomial affine combination $x = \alpha a + \beta b$ in (2.1)*.*

Given the function $f : \mathbb{R}^m \to \mathbb{R}$, let $f_\triangle^{\mathrm{hp}} : \mathbb{R}^m \to \mathbb{R}$ be the function of the hyperplane (in $\mathbb{R}^{m+1}$) passing through the points $(A_j, f(A_j))$ of the graph of f. Applying the affinity of $f_\triangle^{\mathrm{hp}}$ to the combination in (3.2), it follows

$$f_\triangle^{\mathrm{hp}}(P) = \sum_{j=1}^{m+1} \alpha_j f(A_j). \tag{3.5}$$

If we use the convex function f, then we get the hyperplane inequality

$$f(P) \le f_\triangle^{\mathrm{hp}}(P) \ \text{ if } \ P \in \triangle. \tag{3.6}$$

The reverse inequality in (3.2) does not generally hold if $P \notin \triangle$. Examples of this fact is easy to find for two variable convex functions on the triangle.

In this section, we will use the space $\mathbb{X} = C(\triangle)$.

Theorem 3.2. *Let $g_1, \dots, g_m : \triangle \to \mathbb{R}$ be continuous functions such that m-tuple $(g_1(x), \dots, g_m(x)) \in \triangle$ for every $x \in \triangle$. Let $f : \triangle \to \mathbb{R}$ be a continuous convex function.*

Then every positive unital linear functional $L : C(\triangle) \to \mathbb{R}$ *satisfies the inequality*

$$f(L(g_1),\ldots,L(g_m)) \le L(f(g_1,\ldots,g_m)) \le f_{\triangle}^{hp}(L(g_1),\ldots,L(g_m)). \tag{3.7}$$

Proof. Prove the right-hand side of (3.7). First we get the inequality

$$L(f(g_1,\ldots,g_m)) \le L(f_{\triangle}^{\mathrm{hp}}(g_1,\ldots,g_m))$$

using the relation $f \le f_{\triangle}^{\mathrm{hp}}$ of (3.6), and then the equality

$$L(f_{\triangle}^{\mathrm{hp}}(g_1,\ldots,g_m)) = f_{\triangle}^{\mathrm{hp}}(L(g_1),\ldots,L(g_m))$$

relying on the affine equation $f_{\triangle}^{\mathrm{hp}}(x_1,\ldots,x_m) = \sum_{j=1}^{m} k_j x_j + l$ where k_j and l are real constants. □

Taking $e_j(x_1,\ldots,x_m) = x_j$ and applying $g_j = e_j$ to the inequality in (3.7), we get the inequality

$$f(L(e_1),\ldots,L(e_m)) \le L(f) \le f_{\triangle}^{\mathrm{hp}}(L(e_1),\ldots,L(e_m)) \tag{3.8}$$

that can be considered as the functional form of the Jensen and hyperplane inequality, and the Hermite-Hadamard inequality for convex functions with m variables. This will be confirmed with applications that follow.

Let us emphasize once again that $\triangle$ is an m-simplex in $\mathbb{R}^m$ with the vertices $A_1,\ldots,A_{m+1}$.

Jensen's and hyperplane's inequality:

Corollary 3.3. *If* $\sum_{i=1}^{n} p_i P_i$ *is a convex combination from* $\triangle$, *then every continuous convex function* $f : \triangle \to \mathbb{R}$ *satisfies the inequality*

$$f\left(\sum_{i=1}^{n} p_i P_i\right) \le \sum_{i=1}^{n} p_i f(P_i) \le f_{\triangle}^{hp}\left(\sum_{i=1}^{n} p_i P_i\right). \tag{3.9}$$

Convex function behaviour at the simplex edges:

Corollary 3.4. *If convex combinations* $\sum_{i=1}^{n} p_i P_i$ *and* $\sum_{j=1}^{m+1} \alpha_j A_j$ *from* $\triangle$ *have the same center*

$$\sum_{i=1}^{n} p_i P_i = \sum_{j=1}^{m+1} \alpha_j A_j, \tag{3.10}$$

then every continuous convex function $f : \triangle \to \mathbb{R}$ *satisfies the inequality*

$$f\left(\sum_{j=1}^{m+1} \alpha_j A_j\right) \le \sum_{i=1}^{n} p_i f(P_i) \le \sum_{j=1}^{m+1} \alpha_j f(A_j). \tag{3.11}$$

Proof. Taking the summarizing functional

$$L(f) = L_{p_1P_1+\ldots+p_nP_n}^{\mathrm{sum}}(f) = \sum_{i=1}^{n} p_i f(P_i), \tag{3.12}$$

applying the functions e_j and respecting the assumed condition in (3.10) so that

$$(L(e_1), \dots, L(e_m)) = \sum_{i=1}^{n} p_i P_i = \sum_{j=1}^{m+1} \alpha_j A_j, \tag{3.13}$$

and finally using the affinity of $f_{\triangle}^{\mathrm{hp}}$, we get

$$f_{\triangle}^{\mathrm{hp}}\left(L(e_1), \dots, L(e_m)\right) = \sum_{j=1}^{m+1} \alpha_j f(A_j). \tag{3.14}$$

Assuming the convexity of f, and inserting the above convex combinations in (3.8), we obtain the discrete inequality in (3.11). □

The most important consequence of the inequality in (3.8) is the extension of Hermite-Hadamard's inequality to convex functions with more variables:

Corollary 3.5. *Every continuous convex function $f : \triangle \to \mathbb{R}$ satisfies the inequality*

$$f\left(\frac{\sum_{j=1}^{m+1} A_j}{m+1}\right) \le \frac{1}{\mathrm{vol}(\triangle)} \int_{\triangle} f(x_1, \dots, x_m)\, dx_1 \dots dx_m \le \frac{\sum_{j=1}^{m+1} f(A_j)}{m+1} \tag{3.15}$$

where $\mathrm{vol}(\triangle)$ *denotes the volume of the simplex* $\triangle$*.*

Proof. Take advantage of the integrating functional

$$L(f) = L_{\triangle}^{\mathrm{int}}(f) = \frac{1}{\mathrm{vol}(\triangle)} \int_{\triangle} f(x_1, \dots, x_m)\, dx_1 \dots dx_m. \tag{3.16}$$

Its particular values

$$L(e_j) = \frac{1}{\mathrm{vol}(\triangle)} \int_{\triangle} x_j\, dx_1 \dots dx_m \text{ for } j = 1, \dots, m \tag{3.17}$$

are the coordinates of the geometric barycenter of the m-simplex $\triangle$. Then it follows

$$(L(e_1), \dots, L(e_m)) = \frac{1}{m+1} \sum_{j=1}^{m+1} A_j \tag{3.18}$$

and

$$f_{\triangle}^{\mathrm{hp}}\left((L(e_1), \dots, L(e_m)\right) = \frac{1}{m+1} \sum_{j=1}^{m+1} f(A_j). \tag{3.19}$$

Assuming the convexity of f, and inserting the right-hand sides of the equalities in (3.16), (3.18) and (3.19) in (3.8), we get the mixed discrete-integral inequality in (3.15). □

To generalize Corollaries 3.3-3.5, similar to the case of one variable, we introduce a positive continuous function $p:\triangle\to\mathbb{R}$, and continuous functions $g_j:\triangle\to\mathbb{R}$ such that $(g_1(x),\dots,g_m(x))\in\triangle$ for every point $x\in\triangle$. The generalization follows by using the summarizing or integrating positive unital linear functional

$$L_{p,g_1,\dots,g_m}(f)=\frac{L(pf(g_1,\dots,g_m))}{L(p)}. \tag{3.20}$$

In the case of m variables, we have the reduction formula $L_{e_0,e_1,\dots,e_m}=L$ for both functionals.

The generalized Jensen functional for convex functions of m variables for the given functional $L:C(\triangle)\to\mathbb{R}$ can be defined with

$$J_L(f)=L(f)-f(L(e_1),\dots,L(e_m)). \tag{3.21}$$

Relying on the inequality in (3.8), we get the global estimation of the bounds of the functional J_L:

Corollary 3.6. *If $f:\triangle\to\mathbb{R}$ is a continuous convex function, then every positive unital linear functional $L:C(\triangle)\to\mathbb{R}$ satisfies the inequality*

$$\begin{aligned}0\le J_L(f) &\le \left(f_\triangle^{hp}-f\right)(L(e_1),\dots,L(e_m))\\ &\le \max_{(x_1,\dots,x_m)\in\triangle}\left(f_\triangle^{hp}-f\right)(x_1,\dots,x_m).\end{aligned} \tag{3.22}$$

4. Appendix on Extreme Values

In this section we give the calculation of extreme values for the difference of an affine and a convex function. Broader problem of determining the extreme values has been discussed in [6, Propositions 5.1-5.6].

Lemma 4.1. *Let $f:[a,b]\to\mathbb{R}$ be a continuous convex function, and $g:[a,b]\to\mathbb{R}$ be the function defined with $g(x)=kx+l-f(x)$ where k and l are real constants.*

Then the function g has the minimum value

$$\min_{a\le x\le b} g(x)=\min\{g(a)\,,\,g(b)\}\,, \tag{4.1}$$

and the maximum value

$$\max_{a\le x\le b} g(x)=\begin{cases} g(a) & \text{if } f'_-(x)\ge k \text{ for every } x\in(a,b)\\ g(x_0) & \text{if } f'_-(x_0)\le k\le f'_+(x_0) \text{ for some } x_0\in(a,b)\\ g(b) & \text{if } f'_+(x)\le k \text{ for every } x\in(a,b)\end{cases}. \tag{4.2}$$

Additionally, if f is strictly convex and g is not monotone, then a unique number $x_0\in(a,b)$ exists so that

$$g(x_0)=\max_{a\le x\le b} g(x). \tag{4.3}$$

Proof. A function $y = g(x)$ is continuously concave because it is the sum of two continuous concave functions $y = kx + l$ and $y = -f(x)$. The assertion in (4.1) applies since the function g is lower bounded by the chord line connecting endpoints $A(a, g(a))$ and $B(b, g(b))$. The calculating in (4.2) follows from the global maximum property of concave functions. With the additional assumptions the assertion in (4.3) is true because of the strict concavity of the function g. □

We are actually interested in the extreme values of the difference between the chord line and function, especially the maximum case which is elaborated in Figure 1 left.

Corollary 4.2. *Let $f : [a, b] \to \mathbb{R}$ be a continuous convex function, and $g : [a, b] \to \mathbb{R}$ be the function defined with $g(x) = f^{line}_{\{a,b\}}(x) - f(x)$.*

Then the function g has the minimum value

$$\min_{a \le x \le b} g(x) = 0, \tag{4.4}$$

and the maximum value

$$\max_{a \le x \le b} g(x) = \frac{b - x_0}{b - a} f(a) + \frac{x_0 - a}{b - a} f(b) - f(x_0) \tag{4.5}$$

where $x_0 \in (a, b)$ is such that

$$f'_-(x_0) \le \frac{f(b) - f(a)}{b - a} \le f'_+(x_0).$$

Additionally, if f is strictly convex, then the right-hand side of (4.5) takes on the maximum value at the unique number $x_0 \in (a, b)$.

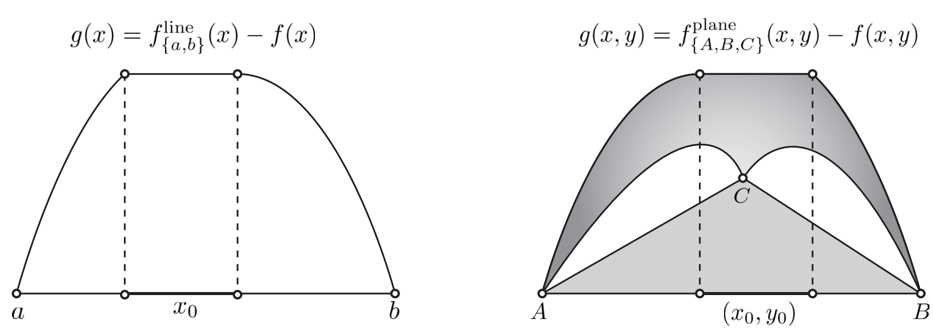

FIGURE 1. Maximum value of a special concave function

If f is a function with more variables, the maximum value of the concave function $g = f^{\text{hp}}_{\triangle} - f$ can be attained at some boundary points of the simplex $\triangle$. Such a case for a function with two variables is shown in Figure 1 right. We also have a concrete example of the function with two variables:

Example 4.3. *Take the triangle with the vertices $A(1,0)$, $B(0,1)$ and $C(0,0)$ in the plane $\mathbb{R}^2$. Observe the convex function $f(x,y)=x^2+y^2$ on the triangle* $\mathrm{conv}\{A,B,C\}$. *The plane passing through the points $A_f(1,0,1)$, $B_f(0,1,1)$ and $C_f(0,0,0)$ of the graph of f has the equation*

$$f^{plane}_{\{A,B,C\}}(x,y)=x+y,$$

so the equation of the difference function $g=f^{plane}_{\{A,B,C\}}-f$ is

$$g(x,y)=x+y-x^2-y^2.$$

The global maximum of the function g on the whole plane $\mathbb{R}^2$ occurs at the point $M(1/2,1/2)$ and has the value $1/2$. The point $M(1/2,1/2)$ belongs to the boundary of the triangle $\mathrm{conv}\{A,B,C\}$ *since it lies on the edge* $\mathrm{conv}\{A,B\}$.

References

[1] P. S. Bullen, D. S. Mitrinović, and P. M. Vasić, *Means and Their Inequalities*, Reidel, Dordrecht, NL, 1988. http://dx.doi.org/10.1007/978-94-017-2226-1

[2] S. S. Dragomir, and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, Melbourne, AU, 2000.

[3] A. E. Farissi, B. Belaidi, and Z. Latreuch, "A variant of Jensen's inequalities", *Malaya Journal of Matematik*, vol. 4, pages 54-60, 2013.

[4] B. Jessen, "Bemærkninger om konvekse Funktioner og Uligheder imellem Middelværdier. I.", *Matematisk tidsskrift. B.*, vol. 2, pages 17-28, 1931.

[5] E. J. McShane, "Jensen's inequality", *Bulletin of the American Mathematical Society*, vol. 43, pages 521-527, 1937. http://dx.doi.org/10.1090/s0002-9904-1937-06588-8

[6] J. Mićić, Z. Pavić, and J. Pečarić, "Some better bounds in converses of the Jensen operator inequality", *Operators and Matrices*, vol. 6, pages 589-605, 2012.

[7] D. S. Mitrinović, and I. B. Lacković, "Hermite and convexity", *Aequationes Mathematicae*, vol. 28, pages 229-232, 1985. http://dx.doi.org/10.1007/bf02189414

[8] C. P. Niculescu, and L. E. Persson, "Old and new on the Hermite-Hadamard inequality", *Real Analysis Exchange*, vol. 29, pages 663-685, 2003.

[9] C. P. Niculescu, and C. I. Spiridon, "New Jensen-type inequalities", arXiv:1207.6877 [math.CA], pages 7, 2012.

[10] Z. Pavić, "Convex combinations, barycenters and convex functions", *Journal of Inequalities and Applications*, vol. 2013, Article 61, pages 13, 2013. http://dx.doi.org/10.1186/1029-242x-2013-61

[11] J. Sándor, "On global bounds for generalized Jensen's inequality", *Annals of Functional Analysis*, vol. 4, pages 18-24, 2013. http://dx.doi.org/10.15352/afa/1399899833

[12] S. Simić, "Sharp global bounds for Jensen's inequality", *Rocky Mountain Journal of Mathematics*, vol. 41, pages 2021-2031, 2011. http://dx.doi.org/10.1216/rmj-2011-41-6-2021

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