Secure Weakly Convex Domination in Graphs

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Abstract

In this paper, we investigate the concept of secure weakly convex domination set of some graphs. We characterized those graphs for which the secure weakly convex domination numbers are 1 and 2. Relations of this parameter with some domination parameters are also observed and a graph is constructed with a preassigned order, weakly convex domination number, secure weakly convex domination number, and secure convex domination number.

Mathematics Subject Classification: 05C69

Keywords: domination, secure domination, weakly convex domination, secure weakly convex domination

1 Introduction

Let $G = (V(G), E(G))$ be a connected undirected graph. For any vertex $v \in V(G)$, the open neighborhood of $v$ is the set $N(v) = \{u \in V(G) : uv \in E(G)\}$ and the closed neighborhood of $v$ is the set $N[v] = N(v) \cup \{v\}$. For a set $X \subseteq V(G)$, the open neighborhood of $X$ is $N(X) = \bigcup_{v \in X} N(v)$ and the closed neighborhood of $X$ is $N[X] = X \cup N(X)$. For any two vertices $u$ and $v$ of $G$, the distance $d_G(u, v)$ is the length of the shortest $u$-$v$ path in $G$. A $u$-$v$
path of length $d_G(u, v)$ is called $u$-$v$ geodesic. A set $C \subseteq V(G)$ is a weakly convex set of $G$ if for every two vertices $u, v \in C$ there exists a $u$-$v$ geodesic whose vertices belongs to $C$, or equivalently, if for every two vertices $u, v \in C$, $d_C(u, v) = d_G(u, v)$. A set $C$ is a convex set of $G$ if for every two vertices $u, v \in C$, the vertex-set of every $u$-$v$ geodesic is contained in $C$.

A set $S$ is a dominating set of $G$ if for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $uv \in E(G)$. The domination number of $G$, denoted by $\gamma(G)$, is the smallest cardinality of a dominating set of $G$. A dominating set of $G$ which is weakly convex (respectively, convex) is called a weakly convex (respectively, convex) dominating set. The weakly convex (respectively, convex) domination number of $G$, denoted by $\gamma_{wcon}(G)$ (respectively, $\gamma_{con}(G)$), is the smallest cardinality of a weakly convex (respectively, convex) dominating set of $G$.

A set $S$ is a secure weakly convex (respectively, secure convex) dominating set of $G$ if $S$ is a weakly convex (respectively, convex) set of $G$ and for every $u \in V(G) \setminus S$, there exists $v \in S$ such that $uv \in E(G)$ and $(S \setminus \{v\}) \cup \{u\}$ is a weakly convex dominating set of $G$. The secure weakly convex (respectively, secure convex) domination number of $G$, denoted by $\gamma_{swc}(G)$ (resp., $\gamma_{scon}(G)$), is the smallest cardinality of a secure weakly convex (respectively, convex) dominating set of $G$.

The concept of weakly convex domination was introduced by Jerzy Topp and is discussed in [3] and [4]. Another domination parameter is the secure domination which was discussed in [1], [2], and [5]. A combination of these two concepts give rise to a new variant of domination called secure weakly convex domination.

**Remark 1.1** Let $G$ be a connected graph of order $n$. Then $1 \leq \gamma_{swc}(G) \leq n$.

### 2 Results

Note that if $S$ is a secure weakly convex dominating set of a connected graph $G$, then $\langle S \rangle$ is connected.

**Proposition 2.1** Let $G$ be a connected graph of order $n \geq 3$ and let $S$ be a secure weakly convex dominating set of $G$.

(i) Every cut-vertex of $G$ is in $S$.

(ii) Every leaf of $G$ is in $S$.

**Proof**: (i) Let $v$ be a cut-vertex of $G$. Then $\langle V(G) \setminus \{v\} \rangle$ consists of at least two components. Let $S$ be a secure weakly convex dominating set of $G$. Suppose $v \notin S$. Since $\langle S \rangle$ is connected, $S$ is contained in some component of $\langle V(G) \setminus \{v\} \rangle$. This implies that $V(G) \setminus S$ contains some vertices in the other components of $\langle V(G) \setminus \{v\} \rangle$. This contradicts the assumption that $S$ is a...
dominating set of $G$. Therefore, $v \in S$.

(ii) Let $v$ be a leaf of $G$. Then $\deg_G(v) = 1$. Let $S$ be a secure weakly convex dominating set of $G$. Suppose that $v \notin S$. Since $S$ is a dominating set of $G$ and $\deg_G(v) = 1$, there exists a unique $u \in S$ such that $uv \in E(G)$. This implies that $\langle (S \setminus \{u\}) \cup \{v\} \rangle$ is not connected. This is a contradiction. Therefore, $v \in S$. □

Note that a star consists of a cut-vertex and leaves; and a path consists of cut-vertices and two leaves. The following results follows from Proposition 2.1

**Corollary 2.2** Let $n \geq 3$ be an integer. Then

(i) $\gamma_{swc}(K_{1,n-1}) = n$.

(ii) $\gamma_{swc}(P_n) = n$.

The next result characterizes a graph $G$ with $\gamma_{swc}(G) = 1$.

**Theorem 2.3** Let $G$ be a connected graph of order $n \geq 2$. Then $G = K_n$ if and only if $\gamma_{swc}(G) = 1$.

*Proof:* Clearly, if $G = K_n$, then $\gamma_{swc}(G) = 1$.

Conversely suppose that $\gamma_{swc}(G) = 1$. Let $S = \{v\}$ be a secure weakly convex dominating set of $G$. Suppose that $G \neq K_n$. Then there exists $u, w \in V(G)$ such that $uw \notin E(G)$. Thus, $(S \setminus \{v\}) \cup \{u\} = \{u\}$, which is not a dominating set of $G$. This is a contradiction. Therefore, $G = K_n$. □

The next result characterizes a graph $G$ with $\gamma_{swc}(G) = 2$.

**Theorem 2.4** Let $G$ be a non-complete connected graph. Then $\gamma_{swc}(G) = 2$ if and only if there exists a non-complete graph $H$ such that $G = K_2 + H$.

*Proof:* Suppose that $\gamma_{swc}(G) = 2$. Let $S = \{u, v\}$ be a secure weakly convex dominating set of $G$. Then $uv \in E(G)$. Define $K_2$ and $H$ by the following: Take $V(K_2) = S$ and $V(H) = V(G) \setminus S$. Then $G = K_2 + H$. Since $G$ is non-complete, it follows that $H$ is non-complete.

Conversely, suppose there exists non-complete graph $H$ such that $G = K_2 + H$. Then $G$ is non-complete. By Theorem 2.3, $\gamma_{swc}(G) \neq 1$ that is, $\gamma_{swc}(G) \geq 2$. Let $S = \{u, v\}$, where $u, v \in V(K_2)$. Then $S$ is a weakly convex set of $G$. By the definition of $K_2 + H$, $S$ is a dominating set of $G$. Let $x \in V(G) \setminus S$. Then $x \in V(H)$ and $ux, vx \in E(G)$. Now, $(S \setminus \{u\}) \cup \{x\} = \{v, x\}$. Since $vx \in E(G)$ and $x$ is arbitrary, $(S \setminus \{u\}) \cup \{x\}$ is a weakly convex dominating set of $G$. This shows that $S$ is a secure weakly convex dominating set of $G$ and $\gamma_{swc}(G) \leq |S| = 2$. Therefore, $\gamma_{swc}(G) = 2$. □
Corollary 2.5 Let $G$ be a non-complete graph and $n \geq 2$. Then $\gamma_{swc}(G + K_n) = 2$.

Proof: Let $H = K_{n-2} + G$. Then $H$ is a non-complete graph. Thus, $G + K_n \cong H + K_2$. By Theorem 2.4, $\gamma_{swc}(G + K_n) = \gamma_{swc}(H + K_2) = 2$. $\square$

Since a secure weakly convex dominating set is a weakly convex dominating set and every secure convex dominating set is a secure weakly convex dominating set, we have

Remark 2.6 Let $G$ be a connected graph. Then $\gamma_{wcon}(G) \leq \gamma_{swc}(G) \leq \gamma_{scon}(G)$.

Theorem 2.7 Given integers $a$, $b$, $c$, and $n$ with $3 \leq a < b < c < n$, there exists a connected graph $G$ such that $|V(G)| = n$, $\gamma_{wcon}(G) = a$, $\gamma_{swc}(G) = b$, $\gamma_{scon}(G) = c$.

Proof: Consider the path $P_{a+1} = [u_1, u_2, ..., u_a, u_{a+1}]$. Let $G$ be a graph obtained from $P_{a+1}$ by adding the edges $u_1v_1$ for $i = 1, 2, ..., b-a-1$, adding the paths $[u_1, w_j, u_3]$ and $[u_2, w_j]$ for $j = 1, 2, ..., c-b$, adding the vertices $z_1, z_2, ..., z_{n-c}$ and forming the complete graph $K_{n-c+2}$, where $V(K_{n-c+2}) = \{z_1, ..., z_{n-c}, u_a, u_{a+1}\}$ (see Figure 1).

Figure 1: A graph $G$ with $\gamma_{wcon}(G) < \gamma_{swc}(G) < \gamma_{scon}(G)$

Then $\{u_1, u_2, ..., u_a\}$ is a weakly convex dominating set of $G$, $\{u_1, u_2, ..., u_a, u_{a+1}\} \cup \{v_1, v_2, ..., u_{b-a-1}\}$ is a secure weakly convex dominating set of $G$, and $\{u_1, u_2, ..., u_a, u_{a+1}\} \cup \{v_1, v_2, ..., u_{b-a-1}\} \cup \{w_1, w_2, ..., w_{c-b}\}$ is a secure convex dominating set of $G$. Hence, $\gamma_{wcon}(G) = a$, $\gamma_{swc}(G) = b$, $\gamma_{scon}(G) = c$. Moreover, $|V(G)| = (a + 1) + (b - a - 1) + (c - b) + (n - c) = n$. $\square$

The next result immediately follows from Theorem 2.7.

Corollary 2.8 For each positive integer $k$, there exists a connected graph $G$ for which $\gamma_{swc}(G) - \gamma_{wcon}(G) = \gamma_{scon}(G) - \gamma_{swc}(G) = k$. 
Corollary 2.9 The domination parameters $\gamma_{\text{swc}}(G)$ and $\gamma_{\text{con}}(G)$ are not comparable.

Proof: Consider a graph $G$ in Theorem 2.7. Then \( \{u_1, u_2, ..., u_a\} \cup \{w_1, w_2, ..., w_{c-b}\} \) is a convex dominating set of $G$. Thus, $\gamma_{\text{con}}(G) = a + c - b$. If $2b \leq a + c$, then $\gamma_{\text{swc}}(G) \leq \gamma_{\text{con}}(G)$. Otherwise, $\gamma_{\text{swc}}(G) > \gamma_{\text{con}}(G)$. This shows that the two domination parameters are not comparable. \( \square \)

References


Received: December 5, 2014; Published: December 22, 2014