Weakly Convexity and Weakly Convex Domination in Graphs

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Abstract
In this paper, we revisit the weakly convexity in graphs and the weakly convex domination in graphs. Some graphs were constructed with a pre-assigned order, diameter, weakly convexity number, convex number, weakly convex domination number and convex domination number.

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1 Introduction
Let $G = (V(G), E(G))$ be a connected graph. For any two vertices $u$ and $v$ of $G$, the distance $d_G(u, v)$ is the length of the shortest $u$-$v$ path in $G$. A $u$-$v$
path of length \(d_G(u, v)\) is called \(u-v\) geodesic. A set \(C \subseteq V(G)\) is called weakly convex in \(G\) if for every two vertices \(u, v \in C\), there exists a \(u-v\) geodesic whose vertices belong to \(C\). A set \(C \subseteq V(G)\) is called convex in \(G\) if for every two vertices \(u, v \in C\), the vertex-set of every \(u-v\) geodesic is contained in \(C\). The weakly convexity number (respectively, convexity number), denoted by \(wcon(G)\) (respectively, \(con(G)\)) of \(G\) is the cardinality of a maximum weakly convex (respectively, convex) proper subset of \(V(G)\). The concept of convexity in graphs is discussed in [1], and [2].

A subset \(S\) of \(V(G)\) is a dominating set of \(G\) if for every \(v \in V(G)\setminus S\), there exists \(u \in S\) such that \(uv \in E(G)\). The domination number \(\gamma(G)\) of \(G\) is the smallest cardinality of a dominating set of \(G\). A set \(S \subseteq V(G)\) is called a weakly convex (respectively, convex) dominating set of \(G\) if it is weakly convex (respectively, convex) and dominating. The weakly convex (respectively, convex) domination number \(\gamma_{wcon}(G)\) (respectively, \(\gamma_{con}(G)\)) of \(G\) is the smallest cardinality of a weakly convex (respectively, convex) dominating set of \(G\). A set \(S \subseteq V(G)\) is a total dominating set of \(G\) if for every \(v \in V(G)\), there exists \(u \in S\) such that \(uv \in E(G)\). The total domination number \(\gamma_t(G)\) is the smallest cardinality of a total dominating set of \(G\).

Lemanska [4] derived relationships between convex and weakly convex domination numbers. Janakiraman and Alphonse [3] established bounds of these parameters and characterized graphs for which bounds are attained. In [5], the weakly convex sets and the weakly convex dominating sets in the corona of graphs were characterized.

## 2 Weakly Convexity in Graphs

**Theorem 2.1** Let \(G\) be a connected graph of order \(n\). Then \(wcon(G) = n-1\) if and only if there exists \(v \in V(G)\) such that \(\langle N(v) \rangle\) is complete or for all \(x, y \in N_G(v)\), \(x \neq y\), \(xy \notin E(G)\), there exists \(w \in V(G)\setminus\{v\}\) with \(x, y \in N_G(w)\).

*Proof:* Suppose \(wcon(G) = n-1\) and let \(S = V(G)\setminus\{v\}\) be a weakly convex set in \(G\). If \(\langle N_G(v) \rangle\) is complete, then we are done. Suppose \(\langle N_G(v) \rangle\) is not complete. Let \(x, y \in N_G(v)\), \(x \neq y\) and \(xy \notin E(G)\). Then there exists an \(x-y\) geodesic \([x, w, y]\) with \(w \in S\). Hence, \(w \neq v\) and \(x, y \in N_G(w)\).

For the converse, suppose that \(v \in V(G)\). Suppose \(\langle N_G(v) \rangle\) is not complete and satisfies the given condition. Pick \(a, b \in S = V(G)\setminus\{v\}\). Let \(P(a, b) = [a_1, a_2, ..., a_k]\) and suppose \(a_r = v\), where \(1 < r < k\). Then \(a_{r-1}, a_{r+1} \notin N_G(v)\) and \(a_{r-1}a_{r+1} \notin E(G)\). By assumption, there exists \(w \in V(G)\setminus\{v\}\) such that \(a_{r-1}, a_{r+1} \in N_G(w)\). Thus, \(P^*(a, b) = [a_1, a_2, ..., a_{r-1}, w, a_{r+1}, ..., a_k]\) is an \(a-b\) geodesic with \(V(P^*(a, b)) \subseteq S\). Hence, \(S\) is weakly convex in \(G\). Therefore,
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For any connected graph $G$, let $\gamma(G)$ denote the domination number of $G$. By Remark 2.2, we have the following result.

**Remark 2.2** For any connected graph $G$ of order $n$, $1 \leq \text{con}(G) \leq wcon(G) \leq n - 1$.

**Theorem 2.3** Given integers $a$, $b$, and $c$ with $4 \leq a < b \leq c$ and $c \leq 2b - a$, there exists a connected graph $G$ such that $\text{diam}(G) = a$, $\text{con}(G) = b$, and $wcon(G) = c$.

**Proof**: Let $P_{a+1}$ be a path $[u_1, u_2, \ldots, u_a, u_{a+1}]$. Let $G$ be a graph obtained from $P_{a+1}$ by adding the paths $[u_1, v_i, u_3]$ for $i = 1, 2, \ldots, b - a$ and the paths $[u_{a-1}, w_j, u_{a+1}]$ for $j = 1, 2, \ldots, c - b$ (see Figure 1). Then $\{u_1, u_2, \ldots, u_a\} \cup \{v_1, v_2, \ldots, v_{b-a}\}$ is a maximum convex proper subset of $G$ and $\{u_1, u_2, \ldots, u_a\} \cup \{v_1, v_2, \ldots, v_{b-a}\} \cup \{w_1, w_2, \ldots, w_{c-b}\}$ is a maximum weakly convex proper subset of $G$. Hence, $\text{diam}(G) = a$, $\text{con}(G) = b$, and $wcon(G) = c$.

Figure 1: A graph $G$ with $\text{diam}(G) < \text{con}(G) \leq wcon(G)$.

**3 Weakly Convex Domination in Graphs**

**Remark 3.1** Let $G$ be a connected graph. Then $\gamma_{wcon}(G) = 1$ if and only if $\gamma(G) = 1$.

The next result is found in [3].

**Remark 3.2** Let $G$ be a connected graph. Then $\gamma(G) \leq \tilde{\gamma}(G) \leq \gamma_{wcon}(G)$.

**Lemma 3.3** Let $G$ be a connected graph. Then $\gamma_{wcon}(G) = 2$ if and only if $\gamma(G) = 2$ and $\gamma(G) \neq 1$.

**Proof**: Suppose that $\gamma_{wcon}(G) = 2$. By Remark 3.1, $\gamma(G) \neq 1$. Thus, $\gamma(G) \geq 2$. By Remark 3.2, $\tilde{\gamma}(G) \geq 2$. Let $C = \{u, v\}$ be a minimum weakly convex dominating set of $G$. Clearly, $uv \in E(G)$. Thus, $C$ is a total dominating set.
of $G$. Hence, $\gamma_t(G) \leq |C| = 2$. Therefore, $\gamma_t(G) = 2$.

Conversely, suppose $\gamma_t(G) = 2$ and $\gamma(G) \neq 1$. By Remark 3.1, $\gamma(wcon)(G) \geq 2$. Let $C = \{u, v\}$ be a minimum total dominating set of $G$. Then $uv \in E(G)$. Hence, $C$ is a weak convex dominating set of $G$. Thus, $\gamma(wcon)(G) \leq |C| = 2$. Therefore, $\gamma(wcon)(G) = 2$. □

The following result is found in [4].

**Remark 3.4** Let $G$ be a connected graph. Then $\gamma(G) \leq \gamma(wcon)(G) \leq \gamma(\text{con})(G)$.

**Theorem 3.5** Given positive integers $a$, $b$, $c$, and $n$ with $3 \leq a \leq b \leq 2a - 1$, $b \leq c$ and $n \geq a + c$, there exists a graph $G$ with $\gamma(G) = a$, $\gamma(wcon)(G) = b$, $\gamma(\text{con})(G) = c$, and $|V(G)| = n$.

**Proof**: Let $P_a$ be the path $[u_1, u_2, ..., u_a]$ and let $H$ be the graph obtained from $P_a$ by adding the edges $x_iy_i$ for $i = 1, 2, ..., a$ and replacing the edges $u_ju_{j+1}$ by the paths $[u_j, v_j, u_{j+1}]$ for $j = 0, 1, 2, ..., b - a$. Let $G$ be the graph obtained from $H$ by adding the edges $y_iu_i$ for $i = 0, 1, 2, ..., n - a - c$ and the paths $[u_1, w_k, u_2]$ for $k = 0, 1, 2, ..., c - b$ (See figure 2). Then $\{u_1, ..., u_a\}$ is a minimum dominating set of $G$, $\{u_1, ..., u_a\} \cup \{v_1, ..., v_{b - a}\}$ is a minimum weakly convex dominating set of $G$, and $\{u_1, ..., u_a\} \cup \{v_1, ..., v_{b - a}\} \cup \{w_1, ..., w_{c - b}\}$ is a minimum convex dominating set of $G$. Therefore, $\gamma(G) = a$, $\gamma(wcon)(G) = b$, $\gamma(\text{con})(G) = c$, and $|V(G)| = n$. □

![Figure 2: A graph $G$ with $\gamma(G) \leq \gamma(wcon)(G) \leq \gamma(\text{con})(G)$](image)

Since $\gamma(wcon)(G) \leq \text{con}(G)$ and from Remarks 2.2 and 3.4, we have

**Remark 3.6** For any connected graph $G$, $\gamma(wcon)(G) \leq \gamma(\text{con})(G) \leq \text{con}(G) \leq \gamma(\text{wcon})(G)$.

**Theorem 3.7** Given integers $a$, $b$, $c$, and $d$ with $5 \leq a \leq b$, $b + 1 \leq c \leq d$, and $d \leq 2c - b - 1$, there exists a connected graph $G$ such that $\gamma(wcon)(G) = a$, $\gamma(\text{con})(G) = b$, $\text{con}(G) = c$, and $\text{wcon}(G) = d$. 
Proof: Let $P_{a+2}$ be a path $[u_1, u_2, ..., u_a, u_{a+1}, u_{a+2}]$. Let $G$ be a graph obtained from $P_{a+2}$ by adding the edges $[u_3, x_k, u_5]$ for $k = 0, 1, 2, ..., b - a$, $[u_a, v_1, u_{a+2}]$ for $i = 0, 1, 2, ..., c - b - 1$, and $[u_1, w_j, u_3]$ for $j = 0, 1, 2, ..., d - c$ (see Figure 3). Then $\{u_2, u_3, ..., u_{a+1}\}$ is a minimum weakly convex dominating set of $G$.

Figure 3: A graph $G$ with $\gamma_{wcon}(G) \leq \gamma_{con}(G) < con(G) \leq wcon(G)$

$\{u_2, u_3, ..., u_a, u_{a+1}\} \cup \{x_1, ..., x_{b-a}\}$ is a minimum convex dominating set of $G$, $\{u_2, u_3, ..., u_{a+2}\} \cup \{x_1, ..., x_{b-a}\} \cup \{v_1, ..., v_{c-b-1}\}$ is a maximum convex proper subset of $G$, and $\{u_2, u_3, ..., u_{a+2}\} \cup \{x_1, x_2, ..., x_{b-a}\} \cup \{v_1, v_2, ..., v_{c-b-1}\} \cup \{w_1, ..., w_{d-c}\}$ is a maximum weakly convex proper subset of $G$. Hence, $\gamma_{wcon}(G) = a$, $\gamma_{con}(G) = b$, $con(G) = c$, and $wcon(G) = d$. \qed

References


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