Instant Blow-up Solutions for
Porous Medium Equation with Sources

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Abstract
In this paper, a porous medium equation with sources in $\mathbb{R}^N$ is considered. We study the solutions of the initial value problem, it is shown that the instant blow-up occurs provided that only nonnegative solutions are considered.

Keywords: porous medium equation, instant blow-up, initial value problem

1 Introduction
In this paper, we consider the initial value problem for a porous medium equation with source as following

$$u_t = \Delta u^m + f(u), \quad (x, t) \in \mathbb{R}^N \times (0, T), \quad (1.1)$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}^N. \quad (1.2)$$

where $m > 1$, $u_0(x)$ is a nonnegative, bounded and continuous function, the nonlinear term $f \in C^1(\mathbb{R})$ and we assume that $f$ is positive, nondecreasing and convex in $(0, \infty)$ and $\int_1^\infty \frac{ds}{f(s)} < \infty$. It is well known that problem (1.1)--(1.2)

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has a unique, nonnegative and bounded solution, defined in some weak sense, at least locally in time [1, 12]. We put

\[ T^* = T^*(u_0) = \sup\{T > 0; u(t) \text{ is bounded and solves Eq. (1.1)-(1.2) in } \mathbb{R}^N \times (0; T)\}, \]

\( T^* \) is called the life span of solutions \( u(t) \). If \( T^* = \infty \) the solutions are global. On the other hand, if \( T^* < \infty \) one has

\[ \lim_{t \to T^*} u(t) = \infty, \]

since otherwise solutions could be extended beyond \( T^* \). When Eq. (1.3) holds we say that the solution blows up in finite time.

When \( f(u) = u^p \) in (1.1), the blow-up and the global existence of solutions are studied by Galaktionov et al. [3], Galaktionov [2], Kawanago [8], and Mochizuki et al. [9]. And the following results are known to hold:

1. Let \( 1 < m < p^* = m + \frac{2}{N} \). Then \( T^* < \infty \) for every nontrivial solution \( u(t) \).

II. Let \( p > m + \frac{2}{N} \). Suppose

\[ u_0(x) \geq E_m(x; t_0; L) \]

for some \( t_0 > 0 \) and some \( L > 0 \) large enough. Then \( T^* < \infty \). Here \( E_m(x; t; L) \) is the Barenblatt solution to the porous media equation

\[ \frac{\partial u}{\partial \ell} = \Delta u_m. \]

III. Let \( p > m + \frac{2}{N} \). Suppose \( u_0 \in L^{\frac{(p-m)N}{2}}(\mathbb{R}^N) \), if \( \|u_0\|_{(p-m)N} \) is sufficiently small, then \( T^* = \infty \) and

\[ \|u(t)\|_{\infty} \leq C t^{\frac{1}{(m+2)/(N-1)}} \text{ as } t \to \infty. \]

The purpose of this article is to prove that \( T^* = 0 \) when initial data \( u_0 \) is growing at the space infinity, i.e., an instant blow-up occurs for the system (1.1)-(1.2). That is to say there is even no local-in-time solution. These problems have been studied by for the Cauchy problem of semilinear equation \( u_t = \Delta u + f(u) \). For other some interesting results about problem (1.1)-(1.2) one can see [4, 5, 6, 7], or [10, 11, 12] when \( f(u) = 0 \). Our results will partly extend theirs to the quasilinear problem (1.1)-(1.2) [1]. Our result is the following theorem.
Theorem 1.1. Suppose that \( f \) is positive, nondecreasing and convex in \((0, \infty)\) and \( \int_1^\infty \frac{ds}{f(s)} < \infty \). If \( u_0 \in C(\mathbb{R}^n) \) is nonnegative and there are a sequence \( x_k \subset \mathbb{R}^n \) with \( |x_k| \to \infty \) as \( k \to \infty \) and a number \( r > 0 \) such that

\[
\lim_{k \to \infty} b_k = \infty \quad \text{with} \quad b_k = \inf \{ u_0(x) : |x - x_k| \leq r \}.
\] (1.3)

Then \( T^* = 0 \). That is, the instant blow-up occurs if only nonnegative solutions are considered.

Remark 1.1. Assumption (1.3) can be relaxed so that \( r = r_k \) depends on \( k \) provided that

\[
\lim_{k \to \infty} \sup \frac{b_k^m}{r_k^2 f(b_k)} < \epsilon
\] (1.4)

with small \( \epsilon > 0 \), say, \( 0 < \epsilon < \epsilon_0 \); the smallness constant \( \epsilon_0 \) depends only on the first eigenvalue of \(-\Delta\) in a unit ball with the Dirichlet boundary condition, thus it only depends on the space dimension \( n \).

2 Proof of our result

Proof. Let \( \lambda_k \) be the principal eigenvalue of \(-\Delta\) with Dirichlet problem in \( B_{R_k}(0) \), and set \( \phi_k(x) \geq 0 \) denote the corresponding positive eigenfunction normalized by \( \int_{B_{R_k}(0)} \phi_k(x)dx = 1 \). By scaling it is easy to see that

\[
\lambda_k = \frac{c}{r_k^2}
\]

for some \( c > 0 \). Consider

\[
G_k(t) = \int_{B(x_k,r_k)} u(x,t)\phi_k(x-x_k)dx.
\]

Then

\[
G_k'(t) = \int_{B(x_k,r_k)} u_t(x,t)\phi_k(x-x_k)dx
\]

\[
= \int_{B(x_k,r_k)} [\Delta u^m(x,t) + f(u(x,t))]\phi_k(x-x_k)dx
\]

Denoted by \( n_k(x) \) the outward unit normal to \( B(0,r_k) \) at \( x \in \partial B(0,r_k) \). We can easily have \( \phi_k = 0 \) and \( \frac{\partial \phi_k}{\partial n_k} \leq 0 \) on \( \partial B(0,r_k) \) with the unit normal vector
Integrating by parts, and using Green’s formula and Jensen’s inequality we have

\[ G_k'(t) \geq \int_{B(x_k, r_k)} u^m(x, t) \Delta \phi_k(x - x_k) \, dx \]
\[ + \int_B (x_k, r_k) f(u_t(x, t)) \phi_k(x - x_k) \, dx \]
\[ \geq -\lambda_k G_k^m(t) + f(G_k(t)). \]

Now we consider the following system of ordinary differential equations

\[ g_k'(t) = -\lambda_k g_k^m(t) + f(g_k(t)), \]
\[ g_k(0) = G_k(0) \geq b_k. \]

Set \( T_{g_k} = \sup\{t \geq 0 : g_k(t) < \infty\} \) and \( T_{G_k} = \sup\{t \geq 0 : G_k(t) < \infty\} \). By a comparison principle, it is easy to see that \( G_k \geq g_k \), thus we have \( T_{g_k} \geq T_{G_k} \).

If \( r_k \) is a constant so that \( \lambda_k = \lambda \) is independ of \( k \), then

\[ T_{G_k} \leq T_{g_k} \leq \int_{b_k}^{\infty} \frac{d\xi}{-\lambda_k \xi^m + f(\xi)} \to 0 \quad \text{as} \quad k \to \infty. \]

This implies that \( T_{G_k} \to 0 \) as \( k \to \infty \). Thus for sufficient large \( k \), \( T_{G_k} < T \). This is a contradiction since \( u \) is continuous in \( \mathbb{R}^N \times [0, T) \).

Next we will discuss the case that \( r_k \to 0 \) as \( k \to \infty \) satisfying (1.4). Now consider the solutions of (1.1) – (1.2) with the initial data \( b_k \). The maximal existence time of the solution denoted by \( T^*(b_k) \) is estimated as

\[ T^*(b_k) = \int_{b_k}^{\infty} \frac{d\xi}{f(\xi)}. \]

Since \( \lim_{k \to \infty} b_k = \infty \), then \( \lim_{k \to \infty} T^*(b_k) = 0 \). Thus the formula

\[ \frac{T^*(b_k)}{T_{g_k}} \geq \frac{\int_{b_k}^{\infty} d\xi / f(\xi)}{\int_{b_k}^{\infty} d\xi / (-\lambda_k \xi^m + f(\xi))}. \quad (2.5) \]

From (1.4), one can assume that there exist \( k_0 \geq 0 \) such that

\[ \frac{b_k^m}{r_k^2 f(b_k)} < \epsilon \]

for \( k \geq k_0 \). Since \( \lambda_k = \frac{c}{r_k^2} \), we see that

\[ \lambda_k b_k^m < c \epsilon f(\xi). \]
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Since \( f \) is positive, nondecreasing and convex in \((0, \infty)\) and \( \int_1^{\infty} \frac{ds}{f(s)} < \infty \), one can obtain
\[
\lambda_k \xi^m < c \epsilon f(\xi)
\]
for \( \xi \geq b_k \). Then
\[
\int_{b_k}^{\infty} \frac{d\xi}{-\lambda \xi^m + f(\xi)} < \int_{b_k}^{\infty} \frac{d\xi}{(1 - c \epsilon) f(\xi)} = \frac{1}{1 - c \epsilon} \int_{b_k}^{\infty} \frac{d\xi}{f(\xi)} \quad (2.6)
\]
From (2.5)–(2.6), we get
\[
\frac{T^*(b_k)}{T_{g_k}} > 1 - c \epsilon > 0
\]
for \( k \geq k_0 \). Thus we obtain
\[
\lim_{k \to \infty} \frac{T^*(b_k)}{T_{g_k}} > 1 - c \epsilon > 0.
\]
Noting that \( \lim_{k \to \infty} T^*(b_k) = 0 \), we see that \( \lim_{k \to \infty} T_{g_k} = 0 \). Again we have \( T_{G_k} \to 0 \) as \( k \to 0 \) which is also a contradiction. So we complete the proof of Theorem 1.1.

References


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