Interpolation of the Five-Point Rectangle by Powers of Polynomial Expressions

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Abstract

The five-point rectangular data array can be interpolated by new polynomial equations and by powers of those equations. New finite difference equations for rectangular arrays are illustrated.

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1. Introduction

There are many operational methods for interpolating four- and five-point rectangles [1-5]. They assume polynomial, exponential, or trigonometric forms. The method of powers for the four-point rectangle has been described [6]. The five-point method follows the four-point methodology. In both cases, the degree of the data, N, is assigned. N can be a small integer like ±2 or it can be a fraction. The methods use the \((x, y) = (-1 .. 1, -1 .. 1)\) coordinate system.
2. Five-point methodology

The five-point rectangle ACEGI is illustrated in Fig. 1. The data at vertices A,C,E,G,I in Fig. 1 are denoted \( P_1, P_2, P_3, P_4, P_5 \), respectively. The group \([P_1,P_2,P_4,P_5]\) has 24 permutations.

Fig. 1 Add T to each member of a permutation and then form 24 equations as in Eq. (5) in [6]. Assign N and solve the 24 equations individually for T within a range of \(-100 \leq T \leq 100\), for example. A unique value of T is usually obtained. Add T to each datum and then take the \( N \)th root of each sum: 
\[
(A+T)^{\frac{1}{N}}, (C+T)^{\frac{1}{N}}, (E+T)^{\frac{1}{N}}, (G+T)^{\frac{1}{N}}, (I+T)^{\frac{1}{N}}
\]
[6]. These numbers are the new A,C,E,G,I, respectively, in Eq. (1) below. Put parentheses around the substituted expression on the right-hand side of Eq. (1), attach the exponent N, and then subtract T. The result is an interpolating equation for the five-point rectangle. Like the four-point method, the five-point method does not apply in every case [6]. R represents an interpolated number.

\[
R = E + (C+I−G−A)(x/4) + (G+I−A−C)(y/4) + (A+I−C−G)(xy/4)
\]

\[
R = (I−A+C−G)^2(4E−A−I−G−C)x^2) / (8((I−A)^2 + (C−G)^2))
\]

\[
R = (I−A−C+G)^2(4E−A−I−G−C)y^2) / (8((I−A)^2 + (C−G)^2))
\]

(1)

For example, suppose the data at A,C,E,G,I are \( A=1^3, C=3^3, E=6^3, G=7^3, I=9^3 \). If the assigned degree of the data is \( N=3 \), the interpolating equation is Eq. (2). If \( N=2 \), the equation is Eq. (3). If \( N=4 \), Eq. (4) is found. If \( N=5 \), Eq. (5) results. If \( N=2.5 \), Eq. (6) appears. If no value of T can be found, change N and try again or expand the range of T. An acceptable interpolating equation reproduces the original data and it renders an acceptable interpolating surface. All numerical coefficients in the equations are rounded.

\[
R = (-0.1000x^2 − 0.9000y^2 + 1.0000x + 3.0000y + 6.0000)^3
\]

\[
R = (-0.1168x^2 + 1.336xy − 1.231y^2 + 2.878x + 9.343y + 14.90)^2 − 6.138
\]

\[
R = (-0.04564x^2 − 0.4706y^2 + 0.4439x + 1.425y + 3.850)^4 − 3.954
\]

\[
R = (-0.02684x^2 − 0.2977y^2 + 0.2588x + 0.8618y + 2.948)^5 − 6.668
\]

\[
R = (-0.1750x^2 − 1.395y^2 + 1.820x + 5.139y + 8.570)^{(5/2)} + 0.9997
\]

(2)

(3)

(4)

(5)

Equations (7)-(9) are alternatives to Eq. (1). Equation (10) is not used by the described five-point method. Most data can also be interpolated directly using Eqs. (1) and (7)-(10). That is easier but less versatile than the power method. P in Eqs. (9) and (10) is \((±)1\).
Interpolation of five-point rectangle

\[
\] (7)

\[
+ ((A+3G+I–C–4E)/8 + (IC–CG–IG+G^2) / (4(I–A+C–G)))x^2
+ ((A+3C+I–G–4E)/8 – (IC–CG–IG+G^2) / (4(I–A+C–G)))y^2
\] (8)

\[
+ ((A+C+G+I–4E)/8 – (P/4)(C–2E+G)\((1/2)\)\((A–2E+I)\((1/2)\))(x^2)
+ ((A+C+G+I–4E)/8 + (P/4)(C–2E+G)\((1/2)\)\((A–2E+I)\((1/2)\))(y^2)
\] (9)

\[
+ (A–C–G+I)^2 x^2 / (8(G+I+A+C–4E+2P((C–2E+G)(A–2E+I))\((1/2)\))
+ (A–C–G+I)^2 y^2 / (8(G+I+A+C–4E–2P((C–2E+G)(A–2E+I))\((1/2)\))
\] (10)

3. Pointwise interpolation equations for five-point rectangles

Equations (48)-(50) in [7] are the products of two expressions. A fourth equation in this category is Eq. (11). All of them vanish when substituted with the linear

\[
\] (11)

numbers (A .. I) = (1 .. 9) in Fig. 1. The figure can be rotated 90° with similar results. For an equation to vanish, one of its factors must vanish. A factor that vanishes by this test becomes an interpolation equation for data arranged by the letters within it. Such equations apply when the x- and y- increments of a two-
variable function have approximately equal effects. That is a severe but not a fatal limitation. Suppose \([E,F,H,I]=[5,6,8,9]\), then Eq. (12) renders \(G=7\). If \([E,F,H,I]=[25,36,64,81]\), then \(G=49\). Both estimates are accurate. If \([E,F,H,I]=[\sin(50^\circ),\sin(60^\circ),\sin(80^\circ),\sin(90^\circ)]\), then \(G\approx 0.9401\). The true value is \(G\approx 0.9397\). The linear numbers causing Eq. (12) to vanish increase monotonically in both the \(x\)- and \(y\)-directions. Trial data should have both of those properties.

\[
(2G-H-F)(E-I)^2 + (E+I-2G)(H-F)^2 = 0
\]  
(12)

Let the rectangle in Fig. 1 be rotated clockwise through \(-90^\circ\). Figure 1 now has \((3,6,9,2,5,8,1,4,7)\) as \(A-I\), respectively. They are monotonic-increasing in the \(x\)-direction but monotonic-decreasing in the \(y\)-direction. The first factor in Eq. (11) now vanishes. This means Eq. (13) is a candidate equation for trial data that are also monotonic-increasing with \((x)\) and monotonic-decreasing with \((y)\).

\[
(2A-D-H)(E-G)^2 + (E+G-2A)(D-H)^2 = 0
\]  
(13)

The method is expanded by using all the factors of Eqs. (48)-(50) in [7]. Four distinct, five-point equations are available. They require five two-dimensional data that are that have uniform spacing in two dimensions. See Fig. 1. In applications, all appropriate equations should be examined. They can render more than one estimate of a missing datum. This method is polynomial-type. It is not suitable for exponential-type data. It has limited accuracy. It can be regarded as a linear method adapted to two dimensions.

There is an error in Section 3 of Ref. [7]. Equation (29) does not belong in the list of polynomial-type formulas. It can be replaced by Eq. (14) below.

\[
2((I-A)^2-(G-C)^2)(B+D+F+H) + ((G-C)^2-(I-A)^2)(A+C+G+I) + 2((I+A)(G-C)^2-(C+G)(I-A)^2) = 0
\]  
(14)

References


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