Hermite-Hadamard Type Inequalities for Differentiable Harmonically $s$-Convex Functions in the Second Sense

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Abstract

In this paper, we give some new Hermite-Hadamard type inequalities, which estimate the difference between the middle and the leftmost terms in the ordinary Hermite-Hadamard inequality, for harmonically $s$-convex functions in the second sense by setting up an integral identity for differentiable functions.

Mathematics Subject Classification: 26A51, 26D15

Keywords: Hermite-Hadamard type inequality, Simpson type inequality, Hölder’s inequality, Harmonically $s$-convexity

1 Introduction

Many inequalities have been established for convex functions but the most famous is the Hermite-Hadamard’s inequality, due to its rich geometrical significance and applications, which is stated as follows: Let $f : I \subseteq R \rightarrow R$ be a convex function and $a, b \in I$ with $a < b$. Then following double inequalities hold:

$$
\left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_{a}^{b} f(x) \, dx \leq \frac{f(a) + f(b)}{2}.
$$

\hspace{1cm} (1)
Hermite-Hadamard’s inequalities for convex, s-convex, m-convex, and GA-convex functions have received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found in [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14] and references therein.

Let us recall some definitions of several kinds of convex functions:

**Definition 1.** Let $I$ be an interval in $R$. Then $f : I \rightarrow R$ is said to be convex on $I$ if the inequality

$$ f (tx + (1 - t)y) \leq tf (x) + (1 - t)f (y) $$

holds, for all $x, y \in I$ and $t \in [0, 1]$.

**Definition 2.** Let $I$ be an interval in $R_+ = (0, \infty)$. A function $f : I \rightarrow R$ is said to be harmonically convex on $I$ if the inequality

$$ f \left( \frac{xy}{tx + (1 - t)y} \right) \leq tf (y) + (1 - t)f (x) $$

holds, for all $x, y \in I$ and $t \in [0, 1]$. If the inequality in (2) is reversed, then $f$ is said to be harmonically concave.

In [1], İmdat İşcan established the following result of the Hermite-Hadamard type for harmonically convex functions:

**Theorem 1.1.** Let $f : I \subseteq R_+ = (0, \infty) \rightarrow R$ be a harmonically convex function on an interval $I$ and $f \in L[a,b]$, where $a, b \in I$ with $a < b$.

$$ f \left( \frac{2ab}{a + b} \right) \leq \frac{ab}{b - a} \int_a^b \frac{f(x)}{x^2} dx \leq \frac{f(a) + f(b)}{2}. $$

(2)

Also, in [1, 2], İmdat İşcan established some new Hermite-Hadamard type inequalities, which estimate the difference between the middle and the rightmost terms in (2), for harmonically convex functions:

**Theorem 1.2.** Let $f : I \subseteq R_+ = (0, \infty) \rightarrow R$ be a differentiable function on the interior $I^0$ of an interval $I$ in $R_+ = (0, \infty)$ and $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is harmonically convex function on $[a, b]$ for $q \geq 1$, then the following inequality holds:

$$ \left| \frac{f(a) + f(b)}{2} - \frac{ab}{b - a} \int_a^b \frac{f(x)}{x^2} dx \right| $$

$$ \leq \frac{ab(b - a)}{2} \lambda_1^{1 - \frac{1}{q}} \left[ \lambda_2 \left| f'(a) \right|^q + \lambda_3 \left| f'(b) \right|^q \right]^{\frac{1}{q}}, $$
where

\[
\begin{align*}
\lambda_1 &= \frac{1}{ab} - \frac{2}{(b-a)^2} \ln \left( \frac{(a+b)^2}{4ab} \right), \\
\lambda_2 &= -\frac{1}{b(b-a)} + \frac{3a + b}{(b-a)^3} \ln \left( \frac{(a+b)^2}{4ab} \right), \\
\lambda_3 &= \frac{1}{a(b-a)} - \frac{3b + a}{(b-a)^3} \ln \left( \frac{(a+b)^2}{4ab} \right) \\
&= \lambda_1 - \lambda_2.
\end{align*}
\]

**Definition 3.** Let \( I \) be an interval in \( \mathbb{R}_+ = (0, \infty) \). A function \( f : I \rightarrow \mathbb{R} \) is said to be harmonically \( s \)-convex in the second sense on \( I \) if the inequality

\[
f \left( \frac{x}{tx + (1-t)y} \right) \leq t^s f(y) + (1-t)^s f(x)
\]

holds, for all \( x, y \in I \) and \( t \in [0,1] \) for some fixed \( s \in (0,1] \). If the inequality in (3) is reversed, then \( f \) is said to be harmonically \( s \)-concave in the second sense.

In this article we consider the following special functions:

**Definition 4.** The hypergeometric function \( \, _2F_1[a, b, c, x] \) is defined for \( |x| < 1 \) by the power series

\[
\, _2F_1[a, b, c, x] = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{x^n}{n!}.
\]

Here \((q)_n\) is the Pochhammer symbol, which is defined by

\[
(q)_n = \begin{cases} 
1, & n = 0 \\
q(q+1)\cdots(q+n-1), & n > 0.
\end{cases}
\]

**Definition 5.** The beta function, also called the Euler integral of the first kind, is a special function defined by

\[
\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt.
\]

In this paper, we give some new Hermite-Hadamard type inequalities, which gives an upper bound for the approximation of the integral average \( \frac{1}{b-a} \int_a^b f(u)du \) by the value \( f\left( \frac{a+b}{2} \right) \), that is, estimate the difference between the middle and the leftmost terms in (1), for harmonically \( s \)-convex functions in the second sense by using an integral identity for differentiable functions.
2 Main results

In order to find some new inequalities of Hermite-Hadamard-like type inequalities connected with the leftmost and and middle parts of (1) for functions whose derivatives are harmonically \( s \)-convex in the second sense, we need the following lemma [11]:

**Lemma 1.** Let \( f : I \subseteq \mathbb{R}^+ = (0, \infty) \to \mathbb{R} \) be a differentiable function on the interior \( I^0 \) of an interval \( I \) such that \( f' \in L([a, b]) \), where \( a, b \in I \) with \( a < b \). Then the following identity

\[
\frac{1}{b - a} \int_a^b f(x)dx - f\left(\frac{a + b}{2}\right) = ab(b - a) \int_0^1 p(t) A_t^3(a, b) f'\left(\frac{ab}{A_t(a, b)}\right) dt
\]

holds, for \( t \in [0, 1] \), where \( A_t(a, b) = (1 - t)a + tb \) and

\[
p(t) = \begin{cases} bt, & t \in [0, \frac{a}{a+b}] \\ a(t-1), & t \in (\frac{a}{a+b}, 1] \end{cases}
\]

**Proof** By the integration by parts, we have

\[
(i) \int_a^{\frac{a}{a+b}} \frac{bt}{A_t^3(a, b)} f'\left(\frac{ab}{A_t(a, b)}\right) dt = \frac{1}{ab(b - a)} \left\{ \frac{1}{b - a} \int_{\frac{a}{a+b}}^b f(x)dx - \frac{1}{2} f\left(\frac{a + b}{2}\right) \right\},
\]

\[
(ii) \int_{\frac{a}{a+b}}^1 \frac{a(t-1)}{A_t^3(a, b)} f'\left(\frac{ab}{A_t(a, b)}\right) dt = \frac{1}{ab(b - a)} \left\{ \frac{1}{b - a} \int_{\frac{a}{a+b}}^a f(x)dx - \frac{1}{2} f\left(\frac{a + b}{2}\right) \right\}.
\]

By the tedious and simple calculations, this is proved.

Now we turn our attention to establish the Hermite-Hadamard type inequalities, which estimate the difference between the middle and the leftmost terms in (1), for harmonically \( s \)-convex functions by using the above lemma.

**Theorem 2.1.** Let \( f : I \subseteq \mathbb{R}^+ = (0, \infty) \to \mathbb{R} \) be a differentiable function on \( I^0 \), the interior of an interval \( I \), such that \( f' \in L([a, b]) \), where \( a, b \in I \) with \( a < b \). If \(|f'|\) is harmonically \( s \)-convex in the second sense on \([a, b]\), then the following inequality holds:

\[
\left| \frac{1}{b - a} \int_a^b f(x)dx - f\left(\frac{a + b}{2}\right) \right|
\leq ab(b - a) \left\{ \left\{ b\mu_{11}(a, b, s) + a\mu_{12}(b, a, s) \right\} |f'(b)|
\right.
\]

\[
+ \left. \left\{ b\mu_{12}(a, b, s) + a\mu_{11}(b, a, s) \right\} |f'(a)| \right\},
\]

(4)
where

\[
\mu_{11}(a, b, s) = \frac{2 \mathcal{F}_1[3, 2 + s, 3 + s, \frac{a-b}{a+b}]}{(s+2)a^{1-s}(a+b)^{2+s}},
\]

\[
\mu_{12}(a, b, s) = \frac{1}{(a-b)^{2+s}} \left[ \frac{1}{a^{1-s}} \left\{ \mathcal{F}_1[1-s, -s, 2-s, \frac{b}{a}] \frac{1}{1-s} \right. \right.
\]

\[
\left. \left. - \mathcal{F}_1[2-s, -s, 3-s, \frac{b}{a}] \right\} \right. \right.
\]

\[
+ \left( a+b \right)^{1-s} \left\{ \left( a+b \right) \mathcal{F}_1[2-s, -s, 3-s, \frac{a+b}{2a}] \frac{1}{2(2-s)b} \right.
\]

\[
\left. \left. - \frac{1}{1-s} \mathcal{F}_1[1-s, -s, 2-s, \frac{a+b}{2a}] \right\} \right].
\]

**Proof** From Lemma 1, we have

\[
\left| \frac{1}{b-a} \int_a^b f(x) \, dx - f \left( \frac{a+b}{2} \right) \right|
\]

\[
= ab(b-a) \left[ b \int_0^{\frac{x+s}{s}} \left| \frac{t}{A_t^s(a,b)} \right| f'' \left( \frac{ab}{A_t^s(a,b)} \right) \right. \right.
\]

\[
\left. \left. \left| f' \left( \frac{ab}{A_t^s(a,b)} \right) \right| \right. \right.
\]

\[
= a \int_{\frac{a+s}{s}}^{1} \left| \frac{1 - t}{A_t^s(a,b)} \right| f' \left( \frac{ab}{A_t^s(a,b)} \right) \right. \right.
\]

\[
\left. \left. \left| f' \left( \frac{ab}{A_t^s(a,b)} \right) \right| \right] \right].
\]

Since \( |f'| \) is harmonically quasi-convex on \([a, b]\), we have

\[
(a) \int_0^{\frac{a+s}{s}} \left| \frac{t}{A_t^s(a,b)} \right| \left| f'' \left( \frac{ab}{A_t^s(a,b)} \right) \right| \, dt
\]

\[
\leq \left( \int_0^{\frac{a+s}{s}} \frac{t}{A_t^s(a,b)} \left\{ t^s |f'(b)| + (1-t)^s |f'(a)| \right\} \right. \, dt
\]

\[
= \left( \int_0^{\frac{a+s}{s}} \frac{t^{s+1}}{A_t^s(a,b)} \right| f'(b) \right| + \left( \int_0^{\frac{a+s}{s}} \frac{t^{s-1}}{A_t^s(a,b)} \right| f'(a) \right|,
\]

\[
\]

\[
(b) \int_{\frac{a+s}{s}}^{1} \left| \frac{1 - t}{A_t^s(a,b)} \right| \left| f' \left( \frac{ab}{A_t^s(a,b)} \right) \right| \, dt
\]

\[
\leq \left( \int_{\frac{a+s}{s}}^{1} \frac{1 - t}{A_t^s(a,b)} \left\{ t^s |f'(b)| + (1-t)^s |f'(a)| \right\} \right. \, dt
\]

\[
= \mu_{11}(b, a, s) |f'(b)| + \mu_{12}(b, a, s) |f'(a)|.
\]

By substituting (6) and (7) in (5), we get the desired result (4).

**Theorem 2.2.** Let \( f : I \subseteq R_+ = (0, \infty) \rightarrow R \) be a differentiable function on the interior \( I^0 \) of an interval \( I \) and \( f' \in L([a,b]), \) where \( a, b \in I \) with \( a < b. \)
If $|f'|^q$ is harmonically $s$-convex in the second sense on $[a, b]$ for $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds, where

\[
\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right| \\
\leq ab(b-a)\left(\frac{1}{s+1}\right)^{\frac{1}{q}} \left[ b\mu_2^s(a, b, p) \right. \\
\times \left\{ \left( \frac{a}{a+b} \right)^{s+1} |f'(b)|^q + \left(1 - \left( \frac{b}{a+b} \right)^{s+1} \right) |f'(a)|^q \right\}^{\frac{1}{q}} \\
+ a\mu_2^s(b, a, p) \\
\times \left\{ \left(1 - \left( \frac{a}{a+b} \right)^{s+1} \right) |f'(b)|^q + \left( \frac{b}{a+b} \right)^{s+1} |f'(a)|^q \right\}^{\frac{1}{q}} \right].
\]

(8)

holds, where

\[
\mu_2(a, b, p) = \frac{a^{1-2p}}{(1+p)(a+b)^{1+p}} \ \ 2 F_1 [3p, 1+p, 2+p, \frac{a-b}{a+b}].
\]

Proof From Lemma 1 and by the Hölder integral inequality, we have

\[
\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right| \\
\leq ab(b-a) \\
\times \left[ b\left( \int_0^{\frac{a}{a+b}} t^p A_t^{3p}(a, b) dt \right)^{\frac{1}{q}} \left( \int_0^{\frac{a}{a+b}} \left| f'\left( \frac{ab}{A_t(a, b)} \right) \right|^q dt \right)^{\frac{1}{q}} \\
+ a\left( \int_0^{1-\left(1-t\right)^p} A_t^{3p}(a, b) dt \right)^{\frac{1}{q}} \left( \int_0^{1-\left(1-t\right)^p} \left| f'\left( \frac{ab}{A_t(a, b)} \right) \right|^q dt \right)^{\frac{1}{q}} \right].
\]

(9)

Note that

\[
(a) \int_0^{\frac{a}{a+b}} t^p A_t^{3p}(a, b) dt = \mu_2(a, b, p),
\]

(10)

\[
(b) \int_0^{1-\left(1-t\right)^p} A_t^{3p}(a, b) dt = \mu_2(b, a, p).
\]

(11)

Since $|f'|^q$ is harmonically $s$-convex in the second sense on $[a, b]$ for $q > 1$,
we have

\[ \int_0^{\frac{a+b}{2}} |f'(\frac{ab}{A_t(a,b)})|^q \, dt \]
\[ \leq \int_0^{\frac{a+b}{2}} \left\{ t^s |f'(b)| + (1-t)^s |f'(a)| \right\} dt \]
\[ \leq \frac{1}{s+1} \left[ \left( \frac{a}{a+b} \right)^{s+1} |f'(b)|^q + \left\{ 1 - \left( \frac{b}{a+b} \right)^{s+1} \right\} |f'(a)|^q \right], \quad (12) \]

\[ \int_{\frac{a+b}{2}}^{a+b} |f'\left(\frac{ab}{A_t(a,b)}\right)|^q \, dt \]
\[ \leq \frac{1}{s+1} \left[ \left\{ 1 - \left( \frac{a}{a+b} \right)^{s+1} \right\} |f'(b)|^q + \left( \frac{b}{a+b} \right)^{s+1} |f'(a)|^q \right]. \quad (13) \]

By substituting (10)-(13) in (9), we get the desired result (8).

**Theorem 2.3.** Let \( f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R} \) be a differentiable function on the interior \( I^0 \) of an interval \( I \) and \( f' \in L([a,b]) \), where \( a, b \in I \) with \( a < b \). If \( |f'|^q \) is harmonically \( s \)-convex in the second sense on \([a,b]\) for \( q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), then the following inequality

\[ \left| \frac{1}{b-a} \int_a^b f(x) \, dx - f\left(\frac{a+b}{2}\right) \right| \]
\[ \leq \left( \frac{1}{1+p} \right)^\frac{1}{p} \left\{ \frac{ab(b-a)}{(a+b)^{1+s}} \right\} \]
\[ \times \left[ a^{1+s} b \left\{ \mu_{31}(a,b,s,q) |f'(b)|^q + \mu_{32}(b,a,s,q) |f'(a)|^q \right\} \right] \]
\[ + ab^{1+s} \left\{ \mu_{32}(a,b,s,q) |f'(b)|^q + \mu_{31}(b,a,s,q) |f'(a)|^q \right\}^{\frac{1}{q}} \]

holds, where

\[ \mu_{31}(a,b,s,q) = \frac{a^{1-3q+s}}{(s+1)(a+b)^{1+s}} \binom{3q}{1+s} \binom{1+s}{2+s} a \]
\[ \mu_{32}(a,b,s,q) = \frac{1}{(1+s)a^{3q}} \binom{2}{1+s} \binom{3q}{2+s} \left( \frac{a-b}{a+b} \right) \]
Proof From Lemma 1 and by the Hölder integral inequality, we have

\[
\left| \frac{1}{b-a} \int_{a}^{b} f(x) dx - f\left(\frac{a+b}{2}\right) \right| \\
\leq ab(b-a) \\
\times \left[ b \left( \int_{0}^{\frac{a}{p+q}} t^p dt \right)^{\frac{1}{p}} \left( \int_{0}^{\frac{a}{p+q}} \frac{1}{A_t^3(a,b)} \left| f' \left( \frac{ab}{A_t(a,b)} \right) \right|^q dt \right)^{\frac{1}{q}} \\
+ a \left( \int_{\frac{a}{p+q}}^{1} (1-t)^p dt \right)^{\frac{1}{p}} \left( \int_{\frac{a}{p+q}}^{1} \frac{1}{A_t^3(a,b)} \left| f' \left( \frac{ab}{A_t(a,b)} \right) \right|^q dt \right)^{\frac{1}{q}} \right]. (15)
\]

Since \(|f'|^q\) is harmonically \(s\)-convex in the second sense on \([a, b]\) for \(q > 1\) with \(\frac{1}{p} + \frac{1}{q} = 1\), we have

\[
(i) \int_{0}^{\frac{a}{p+q}} \frac{1}{A_t^3(a,b)} \left| f' \left( \frac{ab}{A_t(a,b)} \right) \right|^q dt \\
\leq \int_{0}^{\frac{a}{p+q}} \frac{1}{A_t^3(a,b)} \left\{ t^s \left| f'(b) \right|^q + (1-t)^s \left| f'(a) \right|^q \right\} dt
\]

\[
= \int_{0}^{\frac{a}{p+q}} \frac{t^s}{A_t^3(a,b)} \left| f'(b) \right|^q dt + \int_{0}^{\frac{a}{p+q}} \frac{(1-t)^s}{A_t^3(a,b)} \left| f'(a) \right|^q dt \\
= \mu_{31}(a, b, s, q) \left| f'(b) \right|^q + \mu_{32}(b, a, s, q) \left| f'(a) \right|^q, (16)
\]

\[
(ii) \int_{\frac{a}{p+q}}^{1} \frac{1}{A_t^3(a,b)} \left| f' \left( \frac{ab}{A_t(a,b)} \right) \right|^q dt \\
\leq \int_{\frac{a}{p+q}}^{1} \frac{1}{A_t^3(a,b)} \left\{ t^s \left| f'(b) \right|^q + (1-t)^s \left| f'(a) \right|^q \right\} dt
\]

\[
= \mu_{32}(a, b, s, q) \left| f'(b) \right|^q + \mu_{31}(b, a, s, q) \left| f'(a) \right|^q. (17)
\]

Note that

\[
\int_{0}^{\frac{a}{p+q}} t^p dt = \frac{1}{1+p} \left( \frac{a}{a+b} \right)^{1+p}, (18)
\]

\[
\int_{\frac{a}{p+q}}^{1} (1-t)^p dt = \frac{1}{1+p} \left( \frac{b}{a+b} \right)^{1+p}. (19)
\]

By substituting (16)-(19) in (15), we get the desired result (14).
Theorem 2.4. Let $f : I \subseteq \mathbb{R}_+ = (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on the interior $I^0$ of an interval $I$ and $f' \in L([a, b])$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is harmonically $s$-convex in the second sense on $[a, b]$ for $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds, where

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right| \leq ab(b-a) \left[ b\mu_4^\frac{1}{p}(a, b, s, p) \times \left\{ \lambda_{41}(a, b, s, q) |f'(b)|^q + \lambda_{42}(a, b, s, q) |f'(a)|^q \right\}^{\frac{1}{q}} \right. \\
\left. + a\mu_4^\frac{1}{p}(b, a, s, p) \times \left\{ \lambda_{42}(b, a, s, q) |f'(b)|^q + \lambda_{41}(b, a, s, q) |f'(a)|^q \right\}^{\frac{1}{q}} \right]$$

(20)

holds, where

$$\mu_4(a, b, s, p) = \frac{a^{1-3p}}{(3p-1)(b-a)} \left\{ 1 - \left( \frac{2ab}{a+b} \right)^{1-3p} \right\},$$

$$\lambda_{41}(a, b, s, q) = \frac{1}{q+s+1} \left( \frac{a}{a+b} \right)^{q+s+1},$$

$$\lambda_{42}(a, b, s, q) = \beta \left[ \frac{a}{a+b}, 1 + q, 1 + s \right].$$

Proof From Lemma 1 and by the Hölder integral inequality, we have

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right| \leq ab(b-a) \times \left[ \left( \int_0^{\frac{a+b}{2}} \frac{1}{A_4^p(a, b)} dt \right)^{\frac{1}{p}} \left( \int_0^{\frac{a+b}{2}} t^q |f'\left(\frac{ab}{A_4(a, b)}\right)|^q dt \right)^{\frac{1}{q}} \right. \\
\left. + a \left( \int_{\frac{a+b}{2}}^1 \frac{1}{A_4^p(a, b)} dt \right)^{\frac{1}{p}} \left( \int_{\frac{a+b}{2}}^1 (1-t)^q |f'\left(\frac{ab}{A_4(a, b)}\right)|^q dt \right)^{\frac{1}{q}} \right]$$

(21)

Since $|f'|^q$ is harmonically $s$-convex in the second sense on $[a, b]$ for $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have
\( (i) \int_0^\frac{b}{a-b} t^q |f'(\frac{ab}{A_t(a,b)})|^q dt \)
\leq \int_0^\frac{b}{a-b} t^q \left\{ t^q |f'(b)|^q + (1-t)^q |f'(a)|^q \right\} dt
= \int_0^\frac{b}{a-b} t^{q+s} dt |f'(b)|^q + \int_0^\frac{b}{a-b} t^q (1-t)^s dt |f'(a)|^q
= \lambda_{41}(a,b,s,q) |f'(b)|^q + \lambda_{42}(a,b,s,q) |f'(a)|^q, \quad (22)

\( (ii) \int_{\frac{b}{a-b}}^1 (1-t)^q |f'(\frac{ab}{A_t(a,b)})|^q dt \)
\leq \int_{\frac{b}{a-b}}^1 (1-t)^q \left\{ t^q |f'(b)|^q + (1-t)^q |f'(a)|^q \right\} dt
= \lambda_{42}(b,a,s,q) |f'(b)|^q + \lambda_{41}(b,a,s,q) |f'(a)|^q. \quad (23)

By substituting (20)-(21) in (19), we get the desired result (20).

**Theorem 2.5.** Let \( f : I \subseteq R_+ = (0, \infty) \rightarrow R \) be a differentiable function on the interior \( I^0 \) of an interval \( I \) and \( f' \in L([a,b]) \), where \( a,b \in I \) with \( a < b \). If \( |f'|^q \) is harmonically \( s \)-convex in the second sense on \([a,b]\) for \( q \geq 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), then the following inequality holds:

\[
\left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right|
\leq ab(b-a) \times \left[ b \left( \frac{1}{8ab} \right)^\frac{1}{p} \left\{ \lambda_{51}(a,b,s) |f'(b)|^q + \lambda_{52}(a,b,s) |f'(a)|^q \right\}^{\frac{1}{q}} + a \left( \frac{1}{8ab} \right)^\frac{1}{p} \left\{ \lambda_{52}(b,a,s) |f'(b)|^q + \lambda_{51}(b,a,s) |f'(a)|^q \right\}^{\frac{1}{q}} \right], \quad (24)
\]
where
\[
\lambda_{51}(a, b, s) = \frac{1}{(2 + s)a^{1-s}(a + b)^{2+s}} \ 2F_1[3, 2 + s, 3 + s, \frac{a - b}{a + b}],
\]
\[
\lambda_{52}(a, b, s) = \frac{1}{(1 - s)(2 - s)a^{1-s}(a - b)^{2+s}} \left[ \left\{ (2 - s) \ 2F_1[1 - s, -s, 2 - s, \frac{b}{a}] \right. \\
- (1 - s) \ 2F_1[2 - s, -s, 3 - s, \frac{b}{a}] \right] \\
+ \frac{(a + b)^{1-s}}{2^{1-2s}b^{1-s}} \left\{ (1 - s)(a + b) \ 2F_1[2 - s, -s, 3 - s, \frac{a + b}{2a}] \\
- (2 - s)2b \ 2F_1[1 - s, -s, 2 - s, \frac{a + b}{2a}] \right\}. 
\]

Proof From Lemma 1 and by the power mean integral inequality, we have
\[
\left| \frac{1}{b - a} \int_a^b f(x)dx - f\left( \frac{a + b}{2} \right) \right| 
\leq ab(b - a) \\
\times \left[ b \left( \int_0^{\frac{a}{ab}} \frac{t}{A_t^3(a, b)} dt \right)^{\frac{1}{q}} \left( \int_0^{\frac{a}{ab}} \frac{t}{A_t^3(a, b)} \left| f'\left( \frac{ab}{A_t(a, b)} \right) \right|^q dt \right)^{\frac{1}{q}} \\
+ a \left( \frac{1}{8a^2b} \right)^{\frac{1}{q}} \left( \int_0^{\frac{a}{ab}} \frac{1 - t}{A_t^3(a, b)} \left| f'\left( \frac{ab}{A_t(a, b)} \right) \right|^q dt \right)^{\frac{1}{q}} \right].
\]

Since \( |f'|^q \) is harmonically convex on \([a, b]\) for \( q \geq 1 \), we have
\[
(i) \int_0^{\frac{a}{ab}} \frac{t}{A_t^3(a, b)} \left| f'\left( \frac{ab}{A_t(a, b)} \right) \right|^q dt \\
\leq \int_0^{\frac{a}{ab}} \frac{t}{A_t^3(a, b)} \left\{ t^s |f'(b)|^q + (1 - t)^s |f'(a)|^q \right\} dt \\
= \lambda_{51}(a, b, s) |f'(b)|^q + \lambda_{52}(a, b, s) |f'(a)|^q, 
\]
\[
(ii) \int_0^{\frac{a}{ab}} \frac{1 - t}{A_t^3(a, b)} \left| f'\left( \frac{ab}{A_t(a, b)} \right) \right|^q dt \\
\leq \lambda_{52}(b, a, s) |f'(b)|^q + \lambda_{51}(b, a, s) |f'(a)|^q.
\]
By substituting (26) and (27) in (25), we get the desired result (24).

References


Received: June 15, 2014