Fractional Calculus Model for Childhood Diseases and Vaccines

Moustafa El-Shahed

Department of Mathematics, Faculty of Art and Sciences
Qassim University, P.O. Box 3771
Qassim, Unizah 51911, Saudi Arabia

Fatma Abd El-Naby

Department of Mathematics
College of Science and Arts (Female Branch)
Qassim University, P.O. Box 1300
Buraydah 51431, Saudi Arabia

Abstract

This paper deals with the fractional order model for childhood diseases and vaccines. The stability of disease free and positive fixed points is studied. The Adams-Bashforth-Moulton algorithm has been used to solve and simulate the system of differential equations.

Mathematics Subject Classification: 92B05, 93A30, 93C15

Keywords: Childhood diseases; Fractional order; Stability; Numerical method

1. Introduction

Childhood diseases are the most common form of infectious diseases. These are diseases such as measles, mumps, chicken pox, etc., to which children are born susceptible, and usually contract within 5 years. Because young children are in particularly close contact with their peers, at school and play, such diseases can spread quickly. The interest in
modeling the dynamics of infectious diseases dates back a very long time, to the pioneering work of Daniel Bernoulli’s disease statistics from 1760 on smallpox data[4]. Childhood diseases have several attributes which make them well suited for modeling. They have very short incubation and infectious periods and usually confer lifelong immunity. Also, many childhood diseases are mild enough that they do not alter the natural mortality rate. In this case, the underlying demography of the population can be completely disregarded in the model[4, 9]. Mathematical modeling can provide valuable insights into the biological and epidemiological properties of infectious diseases as well as the potential impact of intervention strategies employed by health organizations worldwide. Solutions to systems of differential equations which model disease transmission are of particular use and importance to epidemiologists who wish to study effective means to slow and prevent the spread of disease. Vaccination is a commonly used method for controlling disease, the study of vaccines against infectious disease has been a boon to mankind. For example, the global eradication of smallpox was announced by the World Health Assembly in May 1980. This long-dreaded disease was defeated with a vaccination program. There are now vaccines that are effective in preventing such viral infections as rabies, yellow fever, poliovirus, hepatitis B, parotitis, and encephalitis B. Eventually, vaccines will probably prevent malaria, some forms of heart disease and cancer. Even venereal disease may someday be target of vaccination programs. Vaccines has been very important to every people [12]. The conventional vaccination strategies lead to epidemic eradication if the proportion of the successfully vaccinated individuals is larger than a certain critical value, which is approximately equal to 95 % for measles [4].

In recent decades, the fractional calculus and Fractional differential equations have attracted much attention and increasing interest due to their potential applications in science and engineering [11, 5, 16]. In this paper, we consider the fractional order model for childhood diseases and vaccines. We give a detailed analysis for the asymptotic stability of the model. Adams-Bashforth-Moulton algorithm have been used to solve and simulate the system of differential equations.

2. Model formulation

Let $S$, $I$, $V$ and $R$ denote the densities (or fractions) of susceptible, infected, the density of vaccinees and recovered individuals,respectively. We assume that $\mu$ be the recruitment rate and natural death rate of the population. Let $\beta$ be the transmission rate of disease when susceptible individuals contact with infected individuals and $\gamma$ be the recovery rate of infected individuals. The recovered individuals are assumed to have immunity (so called natural immunity) against the disease. Let $\theta$ be the rate at which susceptible individuals a removed into the vaccination process and $\gamma_1$ be the average rate for them to obtain immunity and move into recovered population. We assume that before obtaining immunity the vaccinees still have the possibility of infection with a disease transmission rate $\beta_1$ while contacting with infected individuals. Using the above assumptions Liu et
al. [14] introduced the following system:
\[
\begin{align*}
\frac{dS}{dt} &= \mu(1 - S) - \beta SI - \theta S, \\
\frac{dV}{dt} &= \theta S - \beta_1 VI - (\gamma_1 + \mu)V, \\
\frac{dI}{dt} &= \beta SI + \beta_1 VI - (\gamma + \mu)I, \\
\frac{dR}{dt} &= \gamma_1 V + \gamma I - \mu R,
\end{align*}
\]
(2.1)
where \(\theta \geq 0\) and all the other parameters are positive. If \(\theta = 0\), which means there are no vaccinations, then \(\lim_{t \to \infty} V(t) = 0\). System (2.1) will reduce to the standard SIR model.

Fractional order models are more accurate than integer-order models as fractional order models allow more degrees of freedom. Fractional differential equations also serve as an excellent tool for the description of hereditary properties of various materials and processes. The presence of memory term in such models not only takes into account the history of the process involved but also carries its impact to present and future development of the process. Fractional differential equations are also regarded as an alternative model to nonlinear differential equations. In consequence, the subject of fractional differential equations is gaining much importance and attention. For some recent work on fractional differential equations, see [5, 11, 16]. Now we introduce fractional order into the ODE model by Liu et al.[14]. The new system is described by the following set of fractional order differential equations:
\[
\begin{align*}
D^\alpha_t S &= \mu(1 - S) - \beta SI - \theta S, \\
D^\alpha_t V &= \theta S - \beta_1 VI - (\gamma_1 + \mu)V, \\
D^\alpha_t I &= \beta SI + \beta_1 VI - (\gamma + \mu)I, \\
D^\alpha_t R &= \gamma_1 V + \gamma I - \mu R,
\end{align*}
\]
(2.2)
where \(D^\alpha_t\) is the Caputo fractional derivative. Because model (2.2) monitors the dynamics of human populations, all the parameters are assumed to be non-negative. Furthermore, it can be shown that all state variables of the model are non-negative for all time \(t \geq 0\) (see, for instance, [4, 9].

**Lemma 2.1.** The closed set \(\Omega = \{(S, V, I, R) \in R^4_+ : S + V + I + R = 1\}\) is positively invariant with respect to model (2.2).

**Proof.** The fractional derivative of the total population, obtained by adding all the equations of model (2.2), is given by
\[
D^\alpha_t N(t) = \mu - \mu N(t)
\]
(2.3)
The solution to Eq. (2.3) is given by \(N(t) = N(0)E_{\alpha,1}(-\mu t^\alpha) + t^\alpha E_{\alpha,\alpha+1}(-\mu t^\alpha)\), where \(E_{\alpha,\beta}\) is the Mittag-Leffler function. Therefore, all solutions of the model with initial conditions in \(\Omega\) remain in \(\Omega\) for all \(t > 0\). Thus, region \(\Omega\) is positively invariant with respect to model (2.2).

In the following, we will study the dynamics of system (2.2).

### 3. Equilibrium Points and Stability

In the following, we discuss the stability of the commensurate fractional ordered dynamical system:
\[
D^\alpha_t x_i = f_i(x_1, x_2, x_3), \quad \alpha \in (0, 1), \quad 1 \leq i \leq 3.
\]
(3.1)
Let $E = (x_1^*, x_2^*, x_3^*)$ be an equilibrium point of system (3.1) and $x_i = x_i^* + \eta_i$, where $\eta_i$ is a small disturbance from a fixed point. Then

$$D_t^a \eta_i = D_t^a x_i = f_i(x_1^* + \eta_1, x_2^* + \eta_2, x_3^* + \eta_3) \approx \eta_1 \frac{\partial f_i(E)}{\partial x_1} + \eta_2 \frac{\partial f_i(E)}{\partial x_2} + \eta_3 \frac{\partial f_i(E)}{\partial x_3}.$$  \hfill (3.2)

System (3.2) can be written as:

$$D_t^a \eta = J \eta,$$  \hfill (3.3)

where $\eta = (\eta_1, \eta_2, \eta_3)^T$ and $J$ is the Jacobian matrix evaluated at the equilibrium points. Using Matignon’s results [15], it follows that the linear autonomous system (3.3) is asymptotically stable if $|\arg(\lambda)| > \frac{\alpha \pi}{2}$ is satisfied for all eigenvalues of matrix $J$ at the equilibrium point $E = (x_1^*, x_2^*, x_3^*)$. If $\Phi(x) = x^3 + a_1 x^2 + a_2 x + a_3$, Let $D(\Phi)$ denote the discriminant of a polynomial $\Phi$, then

$$D(\Phi) = - \begin{vmatrix} 1 & a_1 & a_2 & a_3 & 0 \\ 0 & 1 & a_1 & a_2 & a_3 \\ 3 & 2a_1 & a_2 & 0 & 0 \\ 0 & 3 & a_1 & a_2 & 0 \\ 0 & 0 & 3 & 2a_1 & a_2 \end{vmatrix} = 18a_1a_2a_3 + (a_1a_2)^2 - 4a_3a_1^3 - 4a_2^3 - 27a_3^2.$$

Following [1, 2, 3, 15], we have the proposition.

**Proposition.** One assumes that $E_1$ exists in $R^3_+$. 

1. If the discriminant of $\Phi(x)$, $D(\Phi)$ is positive and Routh-Hurwitz are satisfied, that is, $D(\Phi)>0, a_1 > 0, a_3 > 0, a_1a_2 > a_3$, then $E_1$ is locally asymptotically stable.
2. If $D(\Phi) < 0, a_1 > 0, a_2 > 0, a_1a_2 = a_3, \alpha \in [0, 1)$ then $E_1$ is locally asymptotically stable.
3. If $D(\Phi) < 0, a_1 < 0, a_2 < 0, \alpha > \frac{2}{3}$ then $E_1$ is unstable.
4. The necessary condition for the equilibrium point $E_1$, to be locally asymptotically stable, is $a_3 > 0$.

To evaluate the equilibrium points let

$$D_t^a S = 0, \quad D_t^a V = 0, \quad D_t^a I = 0, \quad D_t^a R = 0.$$  

Then $E_0 = (S_0, V_0, I_0, R_0) = \left( \frac{\mu}{\mu+\theta}, \frac{\theta \mu}{(\mu+\gamma_1)(\mu+\theta)}, 0, \frac{\theta \gamma_1}{(\mu+\gamma_1)(\mu+\theta)} \right)$.

Denote a basic reproduction number [14]

$$R^C = \frac{\beta \mu}{(\mu+\theta)(\mu+\gamma)} + \frac{\beta \theta \mu}{(\mu+\gamma_1)(\mu+\theta)(\mu+\gamma)}.$$  

It means the average new infections produced by one infected individual during his lifespan when the population is at $E_0$. By (2.2), a positive equilibrium $E_1 = (S_1, V_1, I_1, R_1)$ satisfies

$$S_1 = \frac{\mu}{\mu+\theta+\beta I_1}, \quad V_1 = \frac{\theta S_1}{\mu+\gamma_1+\beta I_1}, \quad R_1 = \frac{\theta \gamma_1 S_1}{\mu+\gamma_1+\beta I_1} + \frac{\gamma I_1}{\mu}.$$
and $I_1$ is the positive root of $g(I) = A_1 I^2 + A_2 I + A_3 (1 - R^C)$, where

$$A_1 = (\mu + \gamma) \beta \beta_1 > 0,$$
$$A_2 = (\mu + \gamma) [(\mu + \theta) \beta_1 + (\mu + \gamma_1) \beta] - \beta \beta_1 \mu,$$
$$A_3 = (\mu + \gamma)(\mu + \theta)(\mu + \gamma_1) > 0.$$

The Jacobian matrix $J(E_0)$ for system given in (2.2) evaluated at the disease free equilibrium is as follows:

$$J(E_0) = \begin{pmatrix}
-\mu - \theta & 0 & -\beta S_0 & 0 \\
\theta & -\mu - \gamma_1 & -\beta_1 V_0 & 0 \\
0 & 0 & \beta S_0 + \beta_1 V_0 - \mu - \gamma & 0 \\
0 & \gamma_1 & \gamma & -\mu
\end{pmatrix}.$$  

**Theorem 3.1.** The disease free equilibrium $E_0$ is locally asymptotically stable if $R^C < 1$ and is unstable if $R^C > 1$.

**Proof.** The disease free equilibrium is locally asymptotically stable if all the eigenvalues, $\lambda_i$, $i = 1, 2, 3, 4$ of the Jacobian matrix $J(E_0)$ satisfy the following condition [1, 2, 3, 8, 10, 15]:

$$|\arg(\lambda_i)| > \frac{\alpha \pi}{2}.$$  

(3.4)

The eigenvalues of the characteristic equation of $J(E_0)$ are $\lambda_1 = -\mu$, $\lambda_2 = -(\mu + \gamma_1)$, $\lambda_3 = - (\mu + \theta)$ and $\lambda_4 = (\mu + \gamma)(R^C_0 - 1)$. Hence $E_0$ is locally asymptotically stable if $R^C < 1$ and is unstable if $R^C > 1$.

We now discuss the asymptotic stability of the endemic (positive) equilibrium of the system given by (2.2). The Jacobian matrix $J(E_1)$ evaluated at the endemic equilibrium is given as:

$$J(E_1) = \begin{pmatrix}
-\mu - \theta - \beta I_1 & 0 & -\beta S_1 & 0 \\
\theta & -\mu - \gamma_1 - \beta_1 I_1 & -\beta_1 V_1 & 0 \\
\beta I_1 & \beta_1 I_1 & \beta S_1 + \beta_1 V_1 - \mu - \gamma & 0 \\
0 & \gamma_1 & \gamma & -\mu
\end{pmatrix}.$$  

The characteristic equation of $J(E_1)$ is

$$(\mu + \lambda) \left( \lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3 \right) = 0,$$

where

$$a_1 = \frac{\mu}{S_1} + \frac{\theta S_1}{V_1} > 0, \quad a_2 = \frac{\theta \mu}{V_1} + \beta^2 V_1 I_1 + \beta^2 S_1 I_1 > 0, \quad a_3 = \theta \beta_1 S_1 I_1 + \frac{\theta \beta^2 S_1^2 I_1}{V_1} + \frac{\mu \beta^2 V_1 I_1}{S_1} > 0.$$

4. **Numerical methods and simulations**

Since most of the fractional-order differential equations do not have exact analytic solutions, approximation and numerical techniques must be used. Several analytical and numerical methods have been proposed to solve the fractional order differential equations. For numerical solutions of system (2.2), one can use the generalized Adams-Bashforth-Moulton method. To give the approximate solution by means of this algorithm, consider
the following nonlinear fractional differential equation [6, 7, 13]
\[
D^\alpha_t y(t) = f(t, y(t)), \quad 0 \leq t \leq T,
\]
\[
y^{(k)}(0) = y_0^k, \quad k = 0, 1, 2, \ldots, m - 1, \quad \text{where } m = \lceil \alpha \rceil,
\]
This equation is equivalent to the Volterra integral equation
\[
y(t) = \sum_{k=0}^{m-1} y_0^k \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s)) ds.
\]
(4.1)

Diethelm et al. used the predictor-correctors scheme [6, 7], based on the Adams-Bashforth-Moulton algorithm to integrate Eq. (4.1). By applying this scheme to the fractional-order model for childhood diseases, and setting \( h = \frac{T}{N}, \ t_n = nh, \ n = 0, 1, 2, \ldots, N \in \mathbb{Z}^+, \)
Eq. (4.1) can be discretized as follows [6, 7, 13]:
\[
S_{n+1} = S_0 + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} (\mu(1 - S_j) - \beta S_j I_j - \theta S_j),
\]
\[
V_{n+1} = V_0 + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} (\theta S_j - \beta V_j I_j - (\gamma_1 + \mu)V_j),
\]
\[
I_{n+1} = I_0 + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} (\beta S_j I_j + \beta V_j I_j - (\mu + \gamma) I_j),
\]
\[
R_{n+1} = R_0 + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} (\gamma V_j + \gamma I_j - \mu R_j),
\]
where
\[
a_{j,n+1} = \begin{cases}
\frac{n^{\alpha+1} - (n - \alpha)(n + 1)}{\alpha}, & j = 0, \\
(n - j + 2)^{\alpha+1} + (n - j)^{\alpha+1} - 2(n - j + 1)^{\alpha+1} & 1 \leq j \leq n, \\
1 & j = n + 1,
\end{cases}
\]
\[
b_{j,n+1} = \frac{h^\alpha}{\alpha} ((n - j + 1)^{\alpha} - (n - j)^{\alpha}), \quad 0 \leq j \leq n.
\]

5. Conclusions

In this paper, we consider the fractional order model for childhood diseases and vaccines. We have obtained a stability condition for equilibrium points. We have also given a numerical example and verified our results. One should note that although the equilibrium points are the same for both integer order and fractional order models, the solution of the fractional order model tends to the fixed point over a longer period of time. One also needs to mention that when dealing with real life problems, the order of the system can be determined by using the collected data. The transformation of a classical model into a fractional one makes it very sensitive to the order of differentiation \( \alpha \) : a small change in
α may result in a big change in the final result. From the numerical results in Figure 1, it is clear that the approximate solutions depend continuously on the fractional derivative α.

![Graphs showing approximate solutions S(t), V(t), I(t), and R(t)]

**Figure 1.** The approximate solutions $S(t)$, $V(t)$, $I(t)$, and $R(t)$ are displayed in Figs. a-d, respectively. In each figure three different values of $\alpha$ are considered for $\mu = 1$, $\beta = 10$, $\gamma = 4$, $\gamma_1 = 8$, $\beta_1 = 2$, $\theta = 10$.

**Acknowledgement**

This project was funded by the Deanship of Scientific Research (DSR), Qassim University under grant no. (2053). The authors, therefore, acknowledge with thanks DSR technical and financial support.

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Received: February 15, 2014