Rainbow Connection Numbers of Some Graphs

Syafrizal Sy
Department of Mathematics, Faculty of Mathematics and Natural Science
Universitas Andalas, Kampus Unand Limau Manis, Padang, Indonesia, 25163

Reni Wijaya
Graduate Program of Mathematics, Faculty of Mathematics and Natural Science
Universitas Andalas, Kampus Unand Limau Manis, Padang, Indonesia, 25163

Surahmat
Department of Mathematics Education, Universitas Islam Malang
Jl. MT Haryono 193, Malang 65144, Indonesia

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Abstract

A path in an edge–colored graph is said to be a rainbow path if every edge in the path has different color. An edge colored graph is rainbow connected if there exists a rainbow path between every pair of vertices. The rainbow connection of a graph $G$, denoted by $rc(G)$, is the smallest number of colors required to color the edges of graph such that the graph is rainbow connected.

In this paper, we determine the exact values of $rc(G)$ where $G$ are $G_n, B_n$, and cycle-chain graph $(C_{n_1}, \ldots, C_{n_k})$–path which $C_{n_i}$ is a cycle for every $i = 1, \ldots, k$ with $u_1, u_2, \ldots, u_{k-1}$ are the cut vertices of $G$.

Mathematics Subject Classification: 05C15, 05C40

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1 Introduction

Throughout the paper, all graphs are finite, simple, connected. Let $G$ be such a graph. We write $V(G)$ or $V$ for the vertex set of $G$, and $E(G)$ or $E$ for the edge set $G$. We define a coloring $c : E(G) \rightarrow \{1, 2, \ldots, k\}$, of the edges of $G$, such that the adjacent edges can be colored the same. A $u$-$v$ path $P$ in $G$ is a rainbow path if every edge of $P$ receives different color. The graph $G$ is rainbow-connected (with respect to $c$) if $G$ contains a rainbow $u$-$v$ path for every two vertices $u, v$ in $G$. In this case, the coloring $c$ is called a rainbow coloring of $G$. If $k$ colors are used, then $c$ is a rainbow $k$-coloring. The minimum $k$ such that $G$ has a rainbow $k$-coloring is the rainbow connection number $rc(G)$ of $G$. The rainbow coloring of $G$ using $rc(G)$ colors is called the minimum rainbow coloring of $G$.

The notions of this rainbow coloring was introduced by Chartrand et al. [1] in 2008. Some related to this work are mentioned in the following. Chartrand et al. obtained that $rc(G) = 1$ if and only if $G$ is complete, and that $rc(G) = m$ if and only if $G$ is a tree, as well as that a cycle with $k > 3$ vertices has rainbow connection number $\lceil \frac{k}{2} \rceil$, a triangle has rainbow connection number 1 ([1]). Also notice that, clearly, $rc(G) \geq diam(G)$ where $diam(G)$ denotes the diameter of $G$. Furthermore, Dewi Estetikasari and Syafrizal Sy [2] determined the exact values of rainbow connection for some corona graph.

2 Main Results

These are the main results of the paper.

**Theorem 2.1** For each integer $n$, the rainbow connection of $G$ is $rc(G) = 4$ where $G \cong G_n$ with $n \geq 4$, or $G \cong B_n$ with $n \geq 3$.

**Proof.** We consider two cases.

**Case 1.** For $G \cong G_n$ with $n \geq 4$.

A cycle $C_n$ of length $n \geq 3$ is a connected graph on $n$ vertices in which every vertex has degree two. Let $W_n$ be a wheel of $n + 1$ vertices; namely, a graph consists of a cycle $C_n$ with one additional vertex being adjacent to all vertices of $C_n$. A gear $G_n$ is a wheel graph with a vertex added between each pair of adjacent graph vertices of the outer cycle such that $G_n$ has $2n + 1$ vertices and $3n$ edges. Clearly that $diam(G_n) = 4$. Thus, $rc(G) \geq diam(G_n) = 4$.

Next, we will show that $rc(G) \leq 4$. Consider $V(G_n) = V(C_{2n}) \cup \{v\}$ where $V(C_{2n}) = \{v_1, v_2, \ldots, v_{2n}\}$ is a set of vertices in cycle $C_{2n}$. We define the coloring on $G_n$ by 4-coloring $c : E(G_n) \rightarrow \{1, 2, 3, 4\}$ as follow
By definition of coloring all edges on $G$ of $G$ for every $i$

Stars $S$

A cycle-chain graph ($G$

Therefore, we have $rc(n,)$

Proof $i$ is a cycle for every $i$

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Case 2. For $G \cong B_n$ with $n \geq 3$, clearly that $diam(B_n) = 3$. Let $P := p_i, p, q, p_{i+1}$ be a rainbow path. We consider $G$. A graph $B_n$ consists of two stars $S^1_n$ and $S^2_n$ where the centers are $p$ and $q$, and $p_i$ and $q_i$ are leafs with $i = 1, \ldots, n$, respectively, and for every vertex $p_i$ adjacent to $q_i$.

![Figure 1: Book graph $B_2$](image1)

Next, we define the coloring on $B_n$ by $c : E(B_n) \rightarrow \{1, 2, 3, 4\}$ as follow

$$c(e) = \begin{cases} 
1, & \text{if } e = pq; \\
2, & \text{if } e = pp_i \text{ with } i = 1, \ldots, n; \\
3, & \text{if } e = qq_i \text{ with } i = 1, \ldots, n; \\
4, & \text{if } e = p_i q_i \text{ with } i = 1, \ldots, n.
\end{cases}$$

By definition of coloring of all edges on $G$, we have $rc(G_n) \leq 4$.

Therefore, we have $rc(B_n) = 4$ for $n \geq 3$. □

Let $G \cong (C_{n_1}, \ldots, C_{n_k}) - path$ be a cycle-chain graph which $C_{n_i}$ is a cycle for every $i = 1, \ldots, k$ with $u_1, u_2, \ldots, u_{k-1}$ are the cut vertices of $G$. Fig. 2, A cycle-chain graph $(C_5, C_6, C_5, C_4, C_3)$-path.

Theorem 2.2 For each integer $n_i \geq 3$ and $k \geq 2$, the rainbow connection of $G \cong (C_{n_1}, \ldots, C_{n_k}) - path$ is $rc(G) = \left\lceil \frac{n_1}{2} \right\rceil + \sum_{i=2}^{k} \left\lceil \frac{n_i}{2} \right\rceil$ where $C_{n_i}$ is a cycle on $n_i$ vertices for every $i$.

Proof. We consider cycle-chain graph $G \cong (C_{n_1}, \ldots, C_{n_k}) - path$ which $C_{n_i}$ is a cycle for every $i = 1, \ldots, k$. Let $u_i$ be the cut vertex of $C_{n_i}$ and $C_{n_{i+1}}$.
with $i = 1, 2, \ldots, k - 1$. Thus $U = \{u_1, u_2, \ldots, u_{k-1}\}$ is a set of cut vertices of $G$. Graph $G$ is consist of $\Sigma_{i=1}^{k} n_i$ edges and $\Sigma_{i=1}^{k} n_i - k + 1$ vertices where $k - 1$ vertices are degree 4, and the other vertices are degree 2. By [1], we have $rc(C_{n_i}) = \lceil n_i/2 \rceil$ for every $i = 2, 3, \ldots, k$. As a consequence, since $d(x, u_1) \leq diam(C_{n_1})$ for $x \in C_{n_1}$ then we obtain $rc(G) \geq \lceil n_1/2 \rceil + \Sigma_{i=2}^{k} \lceil n_i/2 \rceil$ where $C_{n_i}$ is a cycle on $n_i$ vertices for every $i$.

Next, we will show that $rc(G) \leq \lceil n_1/2 \rceil + \Sigma_{i=2}^{k} \lceil n_i/2 \rceil$ where $C_{n_i}$ is a cycle on $n_i$ vertices for every $i$. Clearly that, we need the number of colors is $\lceil n_i/2 \rceil$ for every $C_i$ where $i = 1, 2, \ldots, k$. Since $\lceil n_i/2 \rceil - \lfloor n_i/2 \rfloor \leq 1$ then there one edge, say $e$, in $C_{n_i}$ such that $rc(G) \leq \lfloor n_1/2 \rfloor - |\{e\}| + \Sigma_{i=1}^{k} \lceil n_i/2 \rceil$. As a consequence, we obtain $rc(G) \leq \lfloor n_1/2 \rfloor + \Sigma_{i=2}^{k} \lceil n_i/2 \rceil$.

Therefore, we have $rc(G) = \lfloor n_1/2 \rfloor + \Sigma_{i=2}^{k} \lceil n_i/2 \rceil$ where $C_{n_i}$ is a cycle on $n_i$ vertices for every $i$. □

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References


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