Steady 2-D Flow of a Second Grade Fluid in a Symmetrical Diverging Channel of Varying Width

A. M. Siddiqui

Pennsylvania State University, York Campus,
Edgecomb Avenue, 17403, USA

T. Haroon

Department of Mathematics, COMSATS Institute of Information Technology
Park Road, Chak Shehzad Islamabad, 44000, Pakistan

Z. Bano

Department of Mathematics, COMSATS Institute of Information Technology
Park Road, Chak Shehzad Islamabad, 44000, Pakistan

J. H. Smeltzer *

Pennsylvania State University, York Campus
Edgecomb Avenue, 17403, USA

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Abstract

A two dimensional steady flow of an incompressible second grade fluid is investigated in a symmetrically divergent channel of varying width. The non-linear differential equations obtained are solved using ADM. The effect of the dimensionless non-Newtonian elastic and cross-viscosity parameters is shown graphically on axial and normal velocities, shear stress at the curved walls and pressure gradient. It is observed that
the magnitude of the axial velocity and the pressure gradient decreases with an increase in the value of each of these parameters. However, shear stress at the curved wall increases as the value of the elastic parameter increases. Previously published results are presented as a special case of the present study.

Mathematics Subject Classification: Primary 76-xx, Secondary 76Axx

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1 Introduction

The study of fluid flow through convergent or divergent channels has many applications in aerospace, chemical, civil, environmental, mechanical, and biomechanical engineering as well as in understanding the flow in rivers and canals. The steady two dimensional flow of a Newtonian fluid in converging and diverging channels was first studied by Jeffery (1915) and Hamel (1916) independently. They each considered the channel walls to be stationary and the steady flow was caused due to the presence of a source or a sink of fluid volume at the intersection of the walls. They also found a similarity transformation to reduce the Navier-Stokes equations to an ordinary differential equation, and derived some interesting results. The work of Jaffery and Hamel has been expanded upon by many authors [3]-[4]. A complete solution was given by Fraenkal (1962); later, Buitrago (1983) gave more details of the asymmetric solutions [1]. Hooper et al applied the transformation method to obtain the flow of a fluid of variable viscosity [2].

All above mentioned works assumed that the channel walls met at a constant angle. Makinde examined the problem of an incompressible Newtonian fluid flowing through a linearly diverging symmetrical channel [5]. Most of the common fluids in the real world exhibit Newtonian behavior; however, there are also many important classes of fluids that are classified as non-Newtonian. Certain industrial materials such as clay coatings, drilling muds, suspensions, some kinds of oils and greases, polymer melts, elastomers and a number of emulsions have been categorized as non-Newtonian fluids. Non-Newtonian fluids may be classified as: (i) fluids for which the shear stress depends on the shear rate; (ii) fluids for which the relation between the shear stress and shear rate depends on time; or (iii) fluids that possess both elastic and viscous properties, known as visco-elastic or elastico-viscous fluids [1]. It does not seem possible to recommend a single constitutive equation to describe the cases described in (i), (ii) and (iii) because of the great diversity in the physical structure of non-Newtonian fluids. As a result, many different constitutive equations for non-Newtonian fluids have been proposed. Out of these consti-
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The constitutive equation of a second grade fluid is a linear relation between the stress and the first Rivlin-Ericksen tensor, the square of the first Rivlin-Ericksen tensor and the second Rivlin-Ericksen tensor [6]. This constitutive equation has three coefficients and is used for fluids of the visco-elastic type. The governing differential equations of a second grade fluid are of higher order than the Navier-Stokes equations.

Since it is difficult to obtain an exact analytical solution of a nonlinear problem, we may attempt to determine approximate analytic solutions. There are many analytic techniques available to find approximate solutions such as the perturbation method, the Homotopy Perturbation Method (HPM), the Homotopy Analysis Method (HAM), and the Adomian Decomposition Method (ADM) which are all well recognized and widely applied. The Adomian Decomposition Method [7]-[9] in particular drew the attention of researchers due to its ability to solve non-linear ordinary and partial differential equations. A considerable amount of research work has been devoted to the application of this method to a wide class of linear and non-linear ordinary differential equations, partial differential equations and integral equations. Many problems which arise in the applied sciences and engineering can be solved by ADM and the result obtained are often characterized by a higher degree of accuracy. This method provides an analytical solution in the form of an infinite series in which each term can be easily determined. Unlike more traditional methods, the ADM needs no discretization, linearization, spatial transformation, or perturbation and thus has significant advantages over other analytical techniques as well as over numerical methods. As in our case, it is difficult to determine an exact solution for a non-Newtonian fluid, so a truncated number of terms are used in the solution of this problem. The rapid convergence of the series solution obtained by ADM has been discussed by Cherrualt and Adomian [10] and provides insight into the characteristics and behavior of the solution, as in the case with the closed form solution.

The aim of the present work is to investigate the flow through a symmetrical divergent channel of variable width for a non-Newtonian fluid of second grade. To the best of authors knowledge, no attempt has been made so far to discuss the problem of second grade fluid flow through symmetric diverging channel of varying width. The governing highly nonlinear equation representing this problem is solved by Adomian Decomposition Method. In the following section the problem is formulated, solved and then the pertinent results are discussed.
2 Basic Equations

The primary equations that govern the flow of an incompressible second grade fluid in the absence of body forces and thermal effects are:

\[ \text{div} \mathbf{V} = 0, \]
\[ \rho \dot{\mathbf{V}} = \text{div} \mathbf{T}, \]

where \( \rho \) is the constant density, \( \mathbf{V} \) is the velocity vector, \( p \) is the pressure, the dot over \( \mathbf{V} \) denotes the material time derivative and \( \mathbf{T} \) is the Cauchy stress tensor, which is defined as:

\[ \mathbf{T} = -p \mathbf{I} + \mu \mathbf{A}_1 + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2, \]

where \( \mu \) is the coefficient of viscosity, \( \alpha_1 \) and \( \alpha_2 \) are the normal stress modulii and \( \mathbf{A}_1 \) and \( \mathbf{A}_2 \) are the first and second Rivlin-Ericksen tensors respectively, defined as:

\[ \mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T; \quad \mathbf{L} = \nabla \mathbf{V}, \]
\[ \mathbf{A}_2 = \dot{\mathbf{A}}_1 + \mathbf{A}_1 \mathbf{L} + \mathbf{L}^T \mathbf{A}_1, \]

and

\[ (\ast) = (\ast)_t + (\mathbf{V} \cdot \nabla)(\ast), \]

where \( (\ast)_t \) is the partial derivative with respect to \( t \). With the help of equations (3) and (4), the momentum equation (2) becomes:

\[ \rho \dot{\mathbf{V}} = -\nabla p + \mu \nabla^2 \mathbf{V} + (\alpha_1 + \alpha_2) \text{div} \mathbf{A}_1^2 \]
\[ + \quad \alpha_1 \left[ \nabla^2 \mathbf{V}_t + \nabla^2 (\nabla \times \mathbf{V}) \times \mathbf{V} + \nabla \left( \mathbf{V} \cdot \nabla^2 \mathbf{V} + \frac{1}{4} |\mathbf{A}_1|^2 \right) \right]. \]

3 Problem Formulation

Consider the steady fully developed flow of an incompressible second grade fluid through a symmetrical diverging channel of varying width. The Cartesian coordinate system is used in such a way that the \( x \)-axis is in the direction of the flow and the \( y \)-axis is perpendicular to it. For simplicity we assume that the channel is symmetric with respect to the \( x \)-axis. Let \( u \) and \( v \) be the velocity components in the \( x \) and \( y \) directions respectively, and \( y = \pm b(x) \) are the rigid walls (see figure 1). For a two dimensional plane steady flow, we take

\[ \mathbf{V} = [u(x, y), v(x, y)]. \]
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The equations (1) and (6) in component form become:

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (8) \]

\[-\rho v \Omega = -\frac{\partial \hat{p}}{\partial x} - \mu \frac{\partial \Omega}{\partial y} - \alpha_1 v \nabla^2 \Omega, \quad (9)\]

\[\rho u \Omega = -\frac{\partial \hat{p}}{\partial y} + \mu \frac{\partial \Omega}{\partial x} + \alpha_1 u \nabla^2 \Omega, \quad (10)\]

where

\[ \Omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \quad (11) \]

and

\[ \hat{p}(x, y) = p + \frac{\rho}{2}(u^2 + v^2) - \alpha_1 \left\{ u \nabla^2 u + v \nabla^2 v \right\} - \frac{1}{4}(3 \alpha_1 + 2 \alpha_2) A_1^2. \quad (12) \]

The associated boundary conditions are [5]:

Symmetry: \[ \frac{\partial u}{\partial y} = 0, \quad v = 0 \text{ at } y = 0, \quad (13) \]

No-slip: \[ u + v \frac{db}{dx} = 0, \quad v = 0 \text{ at } b(x). \quad (14) \]

The flux across any cross-section of the channel is described as:

\[ Q = \int_{-b(x)}^{b(x)} u dy. \quad (15) \]

The stream function \( \psi \) is introduced as:

\[ u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}. \quad (16) \]

Continuity (8) is satisfied and equations (9) to (12), after simplification, become:

\[ \Omega = -\nabla^2 \psi, \quad (17) \]

\[ \frac{\partial (\psi, \Omega)}{\partial (y, x)} = \nu \nabla^2 \Omega + \frac{\alpha_1}{\rho} \frac{\partial (\psi, \nabla^2 \Omega)}{\partial (y, x)}. \quad (18) \]
The corresponding boundary conditions in terms of $\psi$ become:

$$\frac{\partial^2 \psi}{\partial y^2} = 0, \quad \psi = 0 \quad \text{at} \quad y = 0,$$

and

$$\frac{\partial \psi}{\partial y} - \frac{\partial \psi}{\partial x} \frac{db}{dx} = 0, \quad \psi = Q \quad \text{at} \quad y = b(x).$$

The function $b(x)$ is assumed to depend upon a small parameter $\epsilon$ such that [5]:

$$b(x, \epsilon) = a_0 S \left( \frac{\epsilon x}{a_0} \right) \quad (0 < \epsilon = \frac{a_0}{L} << 1),$$

where $a_0$ is the constant characteristic half width of the channel, $L$ is the constant characteristic length of the channel and $S$ is the function describing the channel wall divergence geometry. In the limit $\epsilon \to 0$, the channel is of constant width and the velocity profile is given by the familiar plane-Poiseuille relation of a parabolic axial velocity profile.

![Figure 1: Schematic diagram of the problem.](image)

Introducing the following dimensionless variables:

$$\tilde{\Omega} = \frac{a_0^2 \Omega}{Q}, \quad \tilde{x} = \frac{\epsilon x}{a_0}, \quad \tilde{y} = \frac{y}{a_0}, \quad \tilde{\psi} = \frac{\psi}{Q},$$

into the equations (17)-(18) and neglecting any terms of order $\epsilon^2$ or higher as well as the bars for simplification, we obtain:

$$\Omega = -\frac{\partial^2 \psi}{\partial y^2},$$

$$\frac{\partial^2 \Omega}{\partial y^2} = Re \left( \frac{\partial (\psi, \Omega)}{\partial (y, x)} - \lambda \frac{\partial (\psi, \frac{\partial^2 \Omega}{\partial y^2})}{\partial (y, x)} \right),$$
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where \( R_e = \frac{Q\epsilon}{\nu} \) is the Reynolds number and \( \lambda = \frac{\alpha_1}{\rho a_0^2} \) is the dimensionless non-Newtonian elastic parameter. Boundary conditions in dimensionless form become:

\[
\frac{\partial^2 \psi}{\partial y^2} = 0, \quad \psi = 0 \quad \text{at} \quad y = 0, \quad (25)
\]

\[
\frac{\partial \psi}{\partial y} = 0, \quad \psi = 1 \quad \text{at} \quad y = S(x). \quad (26)
\]

Equation (23) is a non-linear fourth order partial differential equation subject to the boundary conditions (25) and (26), whose exact solution is difficult to obtain. In this article, we will present an analytical solution using the Adomian decomposition method.

4 Adomian Decomposition Method

In this section, we briefly outline the Adomian decomposition method (ADM) [7]. To give a clear overview of ADM, we first consider:

\[ Fu = g(t), \]

where \( F \) is a differential operator with linear and non-linear terms. The linear term is written as \( Lu + R u \), where \( L \) invertible. To avoid difficult integration we choose \( L \) to be the highest ordered derivative. \( R \) is the remainder of the linear operator. The non-linear term is represented by \( N u \). Thus \( Lu + R u + N u = g \) and we solve to get:

\[ Lu = g - R u - N u. \]

Because \( L \) is invertible, an equivalent expression is:

\[ L^{-1} Lu = L^{-1} g - L^{-1} R u - L^{-1} N u. \]

Solving for \( u \) yields:

\[ u = f - L^{-1} R u - L^{-1} N u, \]

where the function \( f \) represents the terms arising from the integration of the source term \( g \) and using the initial/boundary conditions to evaluate the constants of integration. According to Adomian [7], the solution \( u \) and the non-linear term \( N u \) can be expressed respectively in the form:

\[ u = \sum_{n=0}^{\infty} u_n, \quad N u = \sum_{n=0}^{\infty} A_n, \]
where the $A_n$ are specially generated polynomials for the particular non-linearity, referred to as Adomian polynomials. They are defined in [8] and discussed extensively in [9]. We have now:

$$\sum_{n=0}^{\infty} u_n = f - L^{-1} R \sum_{n=0}^{\infty} u - L^{-1} \sum_{n=0}^{\infty} A_n,$$

so that

$$u_0 = f,$$

$$u_1 = -L^{-1} R u_0 - L^{-1} A_0,$$

$$u_2 = -L^{-1} R u_1 - L^{-1} A_1,$$

$$u_3 = -L^{-1} R u_2 - L^{-1} A_2,$$

and so on. The practical solution will be the n-term approximation:

$$\beta_n = \sum_{i=0}^{n-1} u_i \quad \text{and} \quad \lim_{n \to \infty} \beta_n = \sum_{i=0}^{n-1} u_i = u,$$

by definition. The convergence of this method has been documented by many researchers; for examples see Cherruault and Adomian [10]. In the sequel we apply the decomposition method to our problem.

5 Solution of the Problem

Using equation (23) into (24) and writing the resulting equation in operator form according to the methodology of ADM [7]-[9] we obtain:

$$L_{yyy} \psi = -R_e \left[ \frac{\partial}{\partial(y,x)} \left( \frac{\partial^2 \psi}{\partial y^2}, \psi \right) - \lambda \frac{\partial}{\partial(y,x)} \left( \frac{\partial^4 \psi}{\partial y^4}, \psi \right) \right],$$

(27)

where $L_{yyy} = \frac{\partial^4}{\partial y^4}$. Since $L_{yyy}$ is invertible, applying $L^{-1}$ to the both sides of equation (27) yields:

$$\psi = C_1(x) \frac{y^3}{6} + C_2(x) \frac{y^2}{2} + C_3(x) y + C_4(x)$$

$$- R_e L^{-1} \left[ \frac{\partial}{\partial(y,x)} \left( \frac{\partial^2 \psi}{\partial y^2}, \psi \right) - \lambda \frac{\partial}{\partial(y,x)} \left( \frac{\partial^4 \psi}{\partial y^4}, \psi \right) \right],$$

(28)
where $C_1(x), C_2(x), C_3(x)$ and $C_4(x)$ are arbitrary functions of $x$. To solve equation (28) by ADM, we let:

$$
\psi = \sum_{n=0}^{\infty} \psi_n, \quad \text{and} \quad N\psi = \sum_{n=0}^{\infty} A_n, \quad (29)
$$

where

$$
N\psi = \begin{bmatrix}
\partial \left( \frac{\partial^2 \psi}{\partial y^2}, \psi \right) - \frac{\partial}{\partial (y,x)} \left( \frac{\partial^4 \psi}{\partial y^4}, \psi \right)
\end{bmatrix} . \quad (30)
$$

In view of equations (29)-(30), equation (28) gives:

$$
\sum_{n=0}^{\infty} \psi_n = C_1(x) \frac{y^3}{6} + C_2(x) \frac{y^2}{2} + C_3(x)y + C_4(x) - R_e L^{-1} \sum_{n=0}^{\infty} A_n. \quad (31)
$$

The zeroth component is identified as:

$$
\psi_0 = C_1(x) \frac{y^3}{6} + C_2(x) \frac{y^2}{2} + C_3(x)y + C_4(x), \quad (32)
$$

and the remaining components are identified as the recurrence relation:

$$
\psi_{n+1} = -R_e L^{-1} A_n, \quad n \geq 0, \quad (33)
$$

where $A_n$ are the Adomian polynomials that represent the non-linear term in (30). The first few $A_n$ are found to be:

$$
A_0 = \frac{\partial}{\partial (y,x)} \left( \frac{\partial^2 \psi_0}{\partial y^2}, \psi_0 \right) - \lambda \frac{\partial}{\partial (y,x)} \left( \frac{\partial^4 \psi_0}{\partial y^4}, \psi_0 \right), \quad (34)
$$

$$
A_1 = \frac{\partial}{\partial (y,x)} \left( \frac{\partial^2 \psi_1}{\partial y^2}, \psi_0 \right) + \frac{\partial}{\partial (y,x)} \left( \frac{\partial^2 \psi_0}{\partial y^2}, \psi_1 \right) - \lambda \left( \frac{\partial}{\partial (y,x)} \left( \frac{\partial^4 \psi_1}{\partial y^4}, \psi_0 \right) + \frac{\partial}{\partial (y,x)} \left( \frac{\partial^4 \psi_0}{\partial y^4}, \psi_1 \right) \right), \quad (35)
$$

and

$$
A_2 = \frac{\partial}{\partial (y,x)} \left( \frac{\partial^2 \psi_2}{\partial y^2}, \psi_0 \right) + \frac{\partial}{\partial (y,x)} \left( \frac{\partial^2 \psi_1}{\partial y^2}, \psi_1 \right) + \frac{\partial}{\partial (y,x)} \left( \frac{\partial^2 \psi_0}{\partial y^2}, \psi_2 \right)
$$
\begin{align*}
\lambda \left( \frac{\partial}{\partial (y, x)} \left( \frac{\partial^4 \psi_2}{\partial y^4}, \psi_0 \right) + \frac{\partial}{\partial (y, x)} \left( \frac{\partial^4 \psi_1}{\partial y^4}, \psi_1 \right) + \frac{\partial}{\partial (y, x)} \left( \frac{\partial^4 \psi_0}{\partial y^4}, \psi_2 \right) \right).
\end{align*}

The remaining polynomials are easily generated. Using these polynomials in (33), the first few components can be determined recursively by:

\begin{align*}
\psi_0 &= C_1(x) \frac{y^3}{6} + C_2(x) \frac{y^2}{2} + C_3(x)y + C_4(x), \\
\psi_1 &= -R_e L^{-1} A_0 = L^{-1} \left[ \frac{\partial}{\partial (y, x)} \left( \frac{\partial^2 \psi_0}{\partial y^2}, \psi_0 \right) - \lambda \frac{\partial}{\partial (y, x)} \left( \frac{\partial^4 \psi_0}{\partial y^4}, \psi_0 \right) \right], \\
\psi_2 &= -R_e L^{-1} A_1 = L^{-1} \left[ \frac{\partial}{\partial (y, x)} \left( \frac{\partial^2 \psi_1}{\partial y^2}, \psi_0 \right) + \frac{\partial}{\partial (y, x)} \left( \frac{\partial^2 \psi_0}{\partial y^2}, \psi_1 \right) - \lambda \left( \frac{\partial}{\partial (y, x)} \left( \frac{\partial^4 \psi_1}{\partial y^4}, \psi_0 \right) + \frac{\partial}{\partial (y, x)} \left( \frac{\partial^4 \psi_0}{\partial y^4}, \psi_1 \right) \right) \right], \\
\psi_3 &= -R_e L^{-1} A_2 = L^{-1} \left[ \frac{\partial}{\partial (y, x)} \left( \frac{\partial^2 \psi_2}{\partial y^2}, \psi_0 \right) + \frac{\partial}{\partial (y, x)} \left( \frac{\partial^2 \psi_1}{\partial y^2}, \psi_1 \right) + \frac{\partial}{\partial (y, x)} \left( \frac{\partial^2 \psi_0}{\partial y^2}, \psi_2 \right) \\
&- \lambda \left( \frac{\partial}{\partial (y, x)} \left( \frac{\partial^4 \psi_2}{\partial y^4}, \psi_0 \right) + \frac{\partial}{\partial (y, x)} \left( \frac{\partial^4 \psi_1}{\partial y^4}, \psi_1 \right) + \frac{\partial}{\partial (y, x)} \left( \frac{\partial^4 \psi_0}{\partial y^4}, \psi_2 \right) \right) \right].
\end{align*}

The corresponding boundary conditions, after using (29) in (25) and (26), become:

\begin{align*}
\frac{\partial^2 \psi_0}{\partial y^2} = 0, \quad \psi_0 = 0 \quad \text{on} \quad y = 0, \\
\frac{\partial \psi_0}{\partial y} = 0, \quad \psi_0 = 1 \quad \text{on} \quad y = S(x),
\end{align*}
and for $n \geq 0$:

$$\frac{\partial^2 \psi_n}{\partial y^2} = 0, \quad \psi_n = 0 \quad \text{on} \quad y = 0, \quad (43)$$

and

$$\frac{\partial \psi_n}{\partial y} = 0, \quad \psi_n = 0 \quad \text{on} \quad y = S(x). \quad (44)$$

Solving (37)-(40) subject to boundary conditions (41)-(44), we obtain:

$$\psi_0 = \frac{1}{2} \left( \frac{3y}{S} - \frac{y^3}{S^3} \right), \quad (45)$$

$$\psi_1 = -R_e \left( \frac{3y^7}{280S^7} - \frac{3y^9}{40S^9} + \frac{33y^9}{280S^3} - \frac{3y}{56S} \right) S_x, \quad (46)$$

$$\psi_2 = -R_e^2 \left[ \left( \frac{y^{11}}{4400S^{11}} - \frac{3y^9}{1120S^9} \right) - \frac{51y^7}{4900S^7} - \frac{57y^5}{2800S^5} + \frac{1151y^3}{215600S^3} - \frac{115y}{17248S} \lambda \left( \frac{56S}{9S^9} - \frac{70S}{20S^7} + \frac{41y^3}{70S^5} - \frac{69y}{280S^3} \right) \right] S_x^2$$

$$\psi_3 = -R_e^3 \left[ \left( \frac{y^{15}}{208000S^{15}} - \frac{16493y^9}{3412000S^{13}} + \frac{439093y^3}{1601600S^{11}} - \frac{33939y}{39239200S} \right) S_x^3 \right.$$

$$\left. + \left( \frac{28529y^5}{8624000S^4} - \frac{49559y^3}{1783600S^2} + \frac{67279y}{7134400} \right) S_x S_{xx} + \left( \frac{9y^{15}}{22422400S^{13}} \right) \right] \quad (47)$$

and

$$\psi_3 = -R_e^3 \left[ \left( \frac{y^{15}}{208000S^{15}} - \frac{16493y^9}{3412000S^{13}} + \frac{439093y^3}{1601600S^{11}} - \frac{33939y}{39239200S} \right) S_x^3 \right.$$

$$\left. + \left( \frac{28529y^5}{8624000S^4} - \frac{49559y^3}{1783600S^2} + \frac{67279y}{7134400} \right) S_x S_{xx} + \left( \frac{9y^{15}}{22422400S^{13}} \right) \right] S_x^2$$

$$\left. - \frac{291y^9}{3920S^{11}} + \frac{239y^7}{1225S^9} - \frac{879y^5}{2800S^7} + \frac{190053y^3}{700700S^5} - \frac{5267y}{57200S^3} \right) S_x^3$$

$$\left. \right] \quad (47)$$

$$\left. + \left( \frac{81y^3}{98560S^{11}} - \frac{93y^1}{9856S^{12}} + \frac{2871y^9}{62720S^{10}} - \frac{39y^7}{320S^8} + \frac{1311y^5}{640S^6} - \frac{318939y^3}{1724800S^4} \right) \right] \quad (47)$$
\[
\psi = \frac{1}{2} \left( \frac{3y}{S} - \frac{y^3}{S^3} \right) - R_e \left( \frac{3y^7}{280S^7} - \frac{3y^5}{40S^5} + \frac{33y^3}{280S^3} - \frac{3y}{56S} \right) S_x \\
+ R e^2 \left[ \left( \frac{y^{11}}{440S^{11}} - \frac{3y^9}{1120S^9} + \frac{5y^7}{4900S^7} - \frac{57y^5}{2800S^5} + \frac{4111y^3}{215600S^3} \right) S_x^2 \\
- \frac{115y}{17248S} - \lambda \left( \frac{y}{56S^{11}} - \frac{9y^7}{70S^9} + \frac{9y^5}{20S^7} - \frac{41y^3}{70S^5} - \frac{69y}{280S^3} \right) \right] S_{xx} \\
- \left( \frac{y^{11}}{12320S^{10}} - \frac{y^9}{1120S^8} + \frac{69y^7}{19600S^6} - \frac{3y^5}{400S^4} + \frac{3279y^3}{431200S^2} \right) S_{xx} \\
- R e^3 \left[ \left( \frac{y^{15}}{208000S^{15}} - \frac{23y^{13}}{1601600S^{13}} + \frac{3y^{11}}{1232000S^{11}} - \frac{31y}{19600S^9} \right) S_x^3 \\
+ \frac{184599y^{17}}{6036800S^{17}} - \frac{16493y^5}{4312000S^5} + \frac{439093y^3}{15695680S^3} - \frac{33897y}{3923920S} \right] S_x^3 \\
+ \left( \frac{-y^{15}}{320320S^{14}} + \frac{23y^{13}}{457600S^{12}} - \frac{3y^{11}}{9625S^{10}} + \frac{167y^9}{156800S^8} - \frac{68609y^7}{3018400S^6} \right) y^{15} \\
+ \frac{28529y^9}{8624000S^4} - \frac{49559y^7}{17836000S^2} + \frac{67279y}{7134400} \right) S_x S_{xx} \\
+ \frac{821y^9}{67y^{11}} + \frac{11y^9}{19449y^7} \right) \right] S_{xxx} + \lambda \left( \left( -\frac{103y^{13}}{80080S^{15}} + \frac{117y^{11}}{7700S^{13}} \right) S_x^3 \\
- \frac{291y^9}{3920S^{11}} + \frac{239y^7}{1225S^9} - \frac{879y^5}{2800S^7} + \frac{190053y^3}{700700S^5} - \frac{5267y}{572000S^3} \right) S_x^3 \\
+ \left( \frac{81y^{13}}{98560S^{11}} - \frac{93y^{11}}{62720S^{10}} + \frac{2871y^9}{320S^8} + \frac{39y^7}{3131y^5} - \frac{318939y^3}{1724800S^4} \right)
\]
Making equation (50) dimensionless and ignoring the terms of order \( \epsilon^2 \) and higher order, we obtain:

The dimensionless axial pressure gradient is obtained from equation (9) as:

\[
\frac{\partial p}{\partial x} = \frac{\partial^3 \psi}{\partial y^3} - R_e \left[ \frac{3 \partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^3 \psi}{\partial x^3} \right] + \lambda \left( \frac{\partial \psi}{\partial y} \frac{\partial^3 \psi}{\partial y^3} - \frac{\partial^2 \psi}{\partial x \partial y^2} \right)
\]

(53)

\[
+ (3\lambda + 2\lambda_1) \left( \frac{\partial^3 \psi}{\partial x \partial y^2} \frac{\partial^2 \psi}{\partial y^2} \right)_x.
\]

(54)
where \( \lambda_1 = \frac{\alpha_2}{\rho a_0^2} \) is the dimensionless cross-viscosity parameter.

Using equation (49) the expression for pressure gradient takes the form:

\[
\frac{\partial p}{\partial x} = -\frac{3}{S^3} + R_e \left( \frac{54 S_x}{35 S^3} - \lambda \left( \frac{9 y^2}{2 S^6} - \frac{9}{2 S^4} - \frac{81 y^2}{S^7} + \frac{27 y^4}{4 S^7} - \frac{9 y^2}{S^5} \right) + \frac{9}{4 S^3} S_x \right) + O(R_e^2). \tag{55}
\]

It may be mentioned here that by setting \( \lambda = \lambda_1 = 0 \) in equations (49), (52) and (55) we recover the expression for the stream function \( \psi \), \( \Omega \), shear stress at the curved wall and pressure gradient obtained by Makinde et al [5] as our special case.

### 6 Results and Discussion

An attempt was made to solve the steady two dimensional flow in a diverging channel of varying gap for an incompressible second grade fluid using ADM. The results obtained are evaluated graphically for elastic parameter \( \lambda \) and cross-viscosity parameter \( \lambda_1 \) considering an exponentially diverging geometry, i.e, \( S(x) = e^x \). Figure 2 shows the axial and normal velocity profiles for different values of \( \lambda \). It is observed that the magnitude of the axial velocity decreases with increasing value of \( \lambda \). A parabolic profile is observed for the axial velocity with maximum value at the centre of the channel and minimum value at the walls of the channel. The occurrence of negative axial velocity near the channel walls due to an increase in the elastic parameter \( \lambda \) indicates the possibility of flow reversal near the walls. It is also observed that the normal velocity is unaltered with an increase in the value of \( \lambda \).

The pressure gradient attempting to accelerate the fluid is counteracted by the elastic parameter \( \lambda \) and cross-viscosity parameter \( \lambda_1 \). As a result, a pressure gradient drop is observed in Figure 3. The drop increases with increased values of \( \lambda \) and \( \lambda_1 \). Figure 4 is provided to show the shear stress at the curved wall for varying values of \( \lambda \). It is observed that the magnitude of the shear stress at the curved walls increases with increasing value of \( \lambda \). However the shear stress at the curved wall decreases as the axial distance increases.
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Figure 2: Axial (left) and normal (right) velocity profiles of the flow in an exponentially diverging channel for different values of $\lambda$ at $Re = 0.5$.

Figure 3: Axial pressure gradient of the flow in an exponentially diverging channel for different values of $\lambda$ and $\lambda_1$ at $Re = 0.5$.

7 Conclusion

Two dimensional steady flow of an incompressible second grade fluid in a symmetrical divergent channel of varying width is solved using ADM. Expressions for velocity, shear stress at the curved walls and pressure gradient are calculated in terms of the stream function $\psi$. The results obtained are general since the solution obtained by Makinde et al [5] can be recovered by setting $\lambda = \lambda_1 = 0$. The results obtained are analysed graphically for different values of non-Newtonian parameters $\lambda$ and $\lambda_1$ taking the channel to be diverging exponentially. It is observed that increasing the value of $\lambda$ dampens the axial velocity and axial pressure gradient, whereas a rise in the magnitude of shear stress at the curved wall is observed as the value of $\lambda$ is increased. In general
it can be concluded that λ and λ_1 have influence on the velocity, the shear stress and the pressure gradient. As many industrial fluids are non-Newtonian in nature, it is hoped that this investigation may be helpful to further research in industrial and other fields in which fluid flow through diverging channels is observed.

References


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