Left Centralizers of Semiprime Gamma Rings with Involution

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Abstract

The purpose of this article is to define notions of $I$-involution on $\Gamma$-rings, existence and prove some interesting results:

(i) Let $M$ be a 2-torsion free semiprime $\Gamma$-ring with $I$-involution and satisfying a certain assumption. If $T : M \rightarrow M$ is a Jordan left centralizer on $M$, then $T$ is a left centralizer.

(ii) Let $M$ be a 2-torsion free semiprime $\Gamma$-ring with $I$-involution and $T : M \rightarrow M$ an additive mapping such that

$$T(x\alpha I(x) + I(x)\alpha x) = T(x)\alpha I(x) + I(x)\alpha T(x)$$
holds for $x \in M$ and $\alpha \in \Gamma$. If $M$ has an identity element and $M$ satisfy a certain assumption, then $T(x) = aax$ for any $x \in M$ and some $a \in Z(M)$.

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## 1 Introduction

The notion of a gamma ring was first introduced as an extensive generalization of the concept of a classical ring. From its first appearance, the extensions and generalizations of various important results in the theory of classical rings to the theory of gamma rings have been attracted a wider attentions as an emerging field of research to the modern algebraists to enrich the world of algebra. All over the world, there is a large number of mathematicians are recently engaged to execute more productive and creative results of gamma rings.

Nobusawa[12] first introduced the notion of a $\Gamma$-ring and shown that $\Gamma$-rings, more general than rings. Bernes[1] weakened slightly the conditions in the definition of $\Gamma$-ring in the sense of Nobusawa. Bernes[1], Kyuno[10], Luh[11], Ceven[2], Hoque et al[5,6,8,9], Dey et al[3,4], Ullah et al[14] and others were obtained a large numbers of important basic properties of $\Gamma$-rings in various ways and developed more remarkable results of $\Gamma$-rings. We start with the following necessary definitions.

Let $M$ and $\Gamma$ be additive abelian groups. If there exists an additive mapping $(x, \alpha, y) \rightarrow x\alpha y$ of $M \times \Gamma \times M \rightarrow M$, which satisfies the conditions $(x\alpha y)\beta z=x\alpha (y\beta z)$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, then $M$ is a $\Gamma$-ring. Every ring $M$ is a $\Gamma$-ring with $M=\Gamma$. However a $\Gamma$-ring need not be a ring. Through out the article, we use $M$ as a $\Gamma$-ring.

An additive subgroup $U$ of $M$ is a left (right) ideal of $M$ if $M \Gamma U \subseteq U(UTM \subseteq U)$. If $U$ is both a left and a right ideal, then we say $U$ is an ideal of $M$. The $\Gamma$-ring $M$ is 2-torsion free if $2x=0$ implies $x=0$ for all $x \in M$. An ideal $P_1$ of $M$ is prime if for any ideals $A$ and $B$ of $M$, $A \Gamma B \subseteq P_1$ implies $A \subseteq P_1$ or $B \subseteq P_1$. An ideal $P_2$ of $M$ is semiprime if for any ideal $U$ of $M$, $U \Gamma U \subseteq P_2$ implies $U \subseteq P_2$. $M$ is prime if $a \Gamma M \Gamma b=(0)$ with $a, b \in M$, implies $a=0$ or $b=0$ and semiprime if $a \Gamma M \Gamma a=(0)$ with $a \in M$ implies $a=0$. Furthermore, $M$ is commutative if $x\alpha y=y\alpha x$ for all $x, y \in M$ and $\alpha \in \Gamma$. Moreover, the set $Z(M) = \{x \in M : x\alpha y = y\alpha x$ for all $\alpha \in \Gamma, y \in M \}$ is the centre of $M$. 
In \( M \), \([x, y]_\alpha = x\alpha y - y\alpha x \) is known as the commutator of \( x \) and \( y \) with respect to \( \alpha \), where \( x, y \in M \) and \( \alpha \in \Gamma \). We make the basic commutator identities:

\[
[x\alpha y, z]_\beta = [x, z]_\beta \alpha y + x\alpha [y, z]_\beta;
\]

and \([x, y\alpha z]_\beta = [x, y]_\beta \alpha z + y\alpha [x, z]_\beta\)

for all \( x, y, z \in M \) and \( \alpha, \beta \in \Gamma \). We consider the following assumption:

(A) \( x\alpha y\beta z = x\beta y\alpha z \), for all \( x, y, z \in M \) and \( \alpha, \beta \in \Gamma \).

According to the assumption (A), the above two identities reduce to

\[
[x\alpha y, z]_\beta = [x, z]_\beta \alpha y + x\alpha [y, z]_\beta
\]

and \([x, y\alpha z]_\beta = [x, y]_\beta \alpha z + y\alpha [x, z]_\beta\),

which we extensively use. For existence of such a \( \Gamma \)-ring \( M \), we give the following example.

**Example 1.1** ([2], Example 1.1) Let \( R \) be an associative ring with the unity element 1. Let \( M = M_{1,2}(R) \) and \( \Gamma = \left\{ \begin{pmatrix} n & 1 \\ 0 & 0 \end{pmatrix} : n \text{ is an integer} \right\} \). Then \( M \) is a \( \Gamma \)-ring. A simple verification shows that \( M \) satisfies the assumption (A).

An additive mapping \( T : M \to M \) is a left(right) centralizer if \( T(x\alpha y) = T(x)\alpha y(T(x\alpha y) = x\alpha T(y)) \) holds for all \( x, y \in M \) and \( \alpha \in \Gamma \). A centralizer is an additive mapping which is both a left and a right centralizer. For any fixed \( a \in M \) and \( \alpha \in \Gamma \), the mapping \( T(x) = a\alpha x \) is a left centralizer and \( T(x) = x\alpha a \) is a right centralizer. We shall restrict our attention on left centralizer, since all results of right centralizers are the same as left centralizers. An additive mapping \( D : M \to M \) is a derivation if \( D(x\alpha y) = D(x)\alpha y + x\alpha D(y) \) holds for all \( x, y \in M \), and \( \alpha \in \Gamma \) and is a Jordan derivation if \( D(x\alpha x) = D(x)\alpha x + x\alpha D(x) \) for all \( x \in M \) and \( \alpha \in \Gamma \). An additive mapping \( T : M \to M \) is Jordan left (right) centralizer if \( T(x\alpha x) = T(x)\alpha x(T(x\alpha x) = x\alpha T(x)) \) for all \( x \in M \) and \( \alpha \in \Gamma \). An additive mappings \( T : M \to M \) is a Jordan centralizer if \( T(x\alpha y + y\alpha x) = T(x)\alpha y + y\alpha T(x) \) for all \( x, y \in M \) and \( \alpha \in \Gamma \).

The main goal of this article is to establish some remarkable results involving \( I \)-involution on \( \Gamma \)-rings.

### 2 The Main Results

**Definition 2.1** Let \( M \) be a \( \Gamma \)-ring. Then the mapping \( I : M \to M \) is called an involution if (i) \( II(a) = a \) (ii) \( I(a + b) = I(a) + I(b) \) (iii) \( I(a\alpha b) = I(b)\alpha I(a) \) for all \( a, b \in M \) and \( \alpha \in \Gamma \).
Example 2.1 Let $M$ be a $\Gamma$-ring. Define $M_1 = \{(a, b) : a, b \in M\}$ and $\Gamma_1 = \{ (\alpha, \alpha) : \alpha \in \Gamma \}$. The addition and multiplication on $M_1$ are defined as follows:

$$(a, b) + (c, d) = (a + c, b + d);$$
and $$(a, b)(\alpha, \alpha)(c, d) = (a\alpha c, d\alpha b).$$

Under these addition and multiplication $M_1$ is a $\Gamma_1$-ring.

Define $I : M_1 \rightarrow M_1$ by $I((a, b)) = (b, a)$. Then

$I((a, b)) = I((b, a)) = (a, b).$

$I((a, b) + (c, d)) = I((a + c, b + d))$
$= (b + d, a + c)$
$= (b, a) + (d, c)$
$= I((a, b) + I((c, d)).$

$I((a, b)(\alpha, \alpha)(c, d)) = I((a\alpha c, d\alpha b))$
$= (d\alpha b, a\alpha c)$
$= (d, c)(\alpha, \alpha)(b, a)$
$= I((c, d))(\alpha, \alpha)I((a, b)).$

Therefore, $I$ is an involution of a $\Gamma_1$-ring $M_1$.

First we need the following Lemmas for proving our main results:

Lemma 2.1 Suppose $M$ is a semiprime $\Gamma$-ring with $I$-involution and satisfying the assumption (A). If there exists an element $a \in M$ such that $a\alpha I(x) = a\alpha x$ holds for all $x \in M$ and $\alpha \in \Gamma$, then $a \in Z(M)$.

Proof. First, we replace $x$ by $I(x)\beta y$ in our condition $a\alpha I(x) = a\alpha x$, we have $a\alpha I(I(x)\beta y) = a\alpha I(x)\beta y \Rightarrow a\alpha I(y)\beta x = a\alpha I(x)\beta y \Rightarrow a\alpha y\beta x = a\alpha x\beta y$. Thus we have $a\alpha [x, y]_\beta = 0$. Taking $y = a$ in this relation, we have $a\alpha [x, a]_\beta = 0$. Hence from Lemma 2.3 in [5], $a \in Z(M)$.

Lemma 2.2 Suppose $M$ is a 2-torsion free semiprime $\Gamma$-ring with $I$-involution and satisfying the assumption (A). Let $a \in M$. If $d : M \rightarrow M$ is a derivation such that $[a, d(x)]_\alpha = 0$ for every $x \in M$ and $\alpha \in \Gamma$, then $a \in Z(M)$. 
Proof. Let \( d(x) = [a, x]_\alpha \) for all \( x \in M \) and \( \alpha \in \Gamma \). Then \( d \) is a derivation on \( M \) for, \( d(x\beta y) = [a, x\beta y]_\alpha = x\beta [a, y]_\alpha + [a, x]_\alpha \beta y \), by (A). Hence \( d(x\beta y) = x\beta d(y) + d(x)\beta y \).

Now, \( d^2(x) = d(d(x)) = [a, d(x)]_\alpha = 0 \) by the hypothesis. Since \( d \) is a derivation on \( M \), we have

\[
d^2(x\alpha y) = d(d(x\alpha y)) = d(d(x)\alpha y + x\alpha d(y)) = d^2(x)\alpha y + d(x)\alpha d(y) + d(x)\alpha d(y) + x\alpha d^2(y) = 2d(x)\alpha d(y) + d^2(x)\alpha y + x\alpha d^2(y).
\]

But \( d^2(x\alpha y) = d^2(x) = d^2(y) = 0 \), so, we obtain \( 2d(x)\alpha d(y) = 0 \). Since \( M \) is 2-torsion free, \( d(x)\alpha d(y) = 0 \), for every \( x, y \in M \) and \( \alpha \in \Gamma \). Replacing \( y \) by \( m\beta x \), for every \( m \in M \) and \( \beta \in \Gamma \), we have \( d(x)\alpha d(m\beta x) = 0 \Rightarrow d(x)\alpha d(m)\beta x + d(x)\alpha m\beta d(x) = 0 \). Since \( d(x)\alpha d(m) = 0 \), we have \( d(x)\alpha m\beta d(x) = 0 \), for all \( m \in M \) and \( \alpha, \beta \in \Gamma \). Since \( M \) is semiprime, \( d(x) = 0 \). This shows that \([a, x]_\alpha = 0 \), for all \( x \in M \) and \( \alpha \in \Gamma \). Therefore, \( a \in Z(M) \).

Lemma 2.3 Let \( M \) be a 2-torsion free semiprime \( \Gamma \)-ring with \( I \)-involution and satisfying the assumption (A). Suppose \( d \) be a derivation on \( M \). If \( d(x)\alpha d(y) = 0 \) for all \( x, y \in M \) and \( \alpha \in \Gamma \), then \( d = 0 \).

Proof. The proof is already complete in the above Lemma 2.2.

Theorem 2.1 Suppose \( M \) is a 2-torsion free semiprime \( \Gamma \)-ring with \( I \)-involution and satisfying the assumption (A). Let \( T : M \to M \) be a Jordan left centralizer on \( M \). Then \( T \) is a left centralizer.

Proof. Suppose that \( T \) is a Jordan left centralizer on \( M \). Then we have,

\[
T(x\alpha I(x)) = T(x)\alpha I(x) \tag{1}
\]

If we replace \( x \) by \( x + y \), we obtain

\[
T(x\alpha I(y) + y\alpha I(x)) = T(x)\alpha I(y) + T(y)\alpha I(x) \tag{2}
\]

Replacing \( y \) by \( I(x) \) in the above relation, we have

\[
T(x\alpha x + I(x)\alpha I(x)) = T(x)\alpha x + T(I(x))\alpha I(x) \tag{3}
\]

\[
\Rightarrow T(x\alpha x) - T(x)\alpha x + T(I(x)\alpha I(x)) - T(I(x))\alpha I(x) = 0
\]

\[
\Rightarrow A(x) + A(I(x)) = 0 \tag{4}
\]
where $A(x) = T(x\alpha x) - T(x\alpha x)$ and $A(I(x)) = T(I(x)\alpha I(x)) - T(I(x)\alpha I(x))$.

Putting $y = x\beta I(y) + y\beta I(x)$ in (2) and using (A), we have

\[
T(x\alpha y\beta I(x) + x\beta I(y)\alpha I(x)) = -T(x\alpha x)\beta I(y) + T(x\alpha x)\beta I(x)
+ T(x\beta x\beta I(x) + T(x)\beta I(y)\alpha I(x)
= (T(x\alpha x) - T(x\alpha x)\beta I(y) + T(x\alpha x)\beta I(x)
+ T(x)\beta I(y)\alpha I(x)
\]

\[
T(x\alpha (y + I(y))\beta I(x)) = -A(x)\beta I(y) + T(x)\alpha y\beta I(x) + T(x)\beta I(y)\alpha I(x)
\]

Putting $y = y - I(y)$ in the relation (5) and using (A), we have

\[
T(xy\alpha \beta I(x)) = -A(x)\beta I(y) + T(x)\alpha y\beta I(x)
\]

\[A(x)\beta y = A(x)\beta I(y)\]  \hspace{1cm} (6)

Hence from Lemma-2.1, $A(x) \in Z(M)$. Replacing $y$ by $I(y)$ in (2), we have

\[
T(x\alpha y + I(y)\alpha I(x)) = T(x)\alpha y + T(I(y))\alpha I(x)
\]

Putting $y = x\beta y$ in (7), we have

\[
T(x\alpha x\beta y + I(y)\beta I(x)\alpha I(x)) = T(x)\alpha x\beta y + T(I(y))\beta I(x)\alpha I(x)
\]

Putting $x = x\beta x$ in (7), we have

\[
T(x\beta x\alpha y + I(y)\alpha I(x)\beta I(x)) = T(x)\beta x\alpha y + T(I(y))\alpha I(x)\beta I(x)
\]

Subtracting (8) from (9), we obtain

\[
A(x)\beta y + (T(I(y))\alpha I(x) - T(I(y)\alpha I(x)))\beta I(x) = 0
\]

For $y = x$ in (10), we have

\[
A(x)\beta x + (T(I(x))\alpha I(x) - T(I(x)\alpha I(x)))\beta I(x) = 0
\]

\[
\Rightarrow A(x)\beta x - A(I(x))\beta I(x) = 0
\]

Hence, using (4), we obtain

\[
A(x)\beta (x + I(x)) = 0
\]  \hspace{1cm} (11)
In particular \( y = x \) in (6), we obtain
\[
A(x)\beta(x - I(x)) = 0 \tag{12}
\]
Hence from (11) and (12), by 2-torsion freeness, we obtain
\[
A(x)\beta x = 0 \tag{13}
\]
Since \( A(x) \in Z(M) \), for all \( x \in M \), we have
\[
x\beta A(x) = 0 \tag{14}
\]
By linearization of (13) gives
\[
A(x)\beta y + A(y)\beta x + B(x,y)\beta x + B(x,y)\beta y = 0 \tag{15}
\]
where \( B(x,y) = T(x\alpha y + y\alpha x) - T(x)\alpha y - T(y)\alpha x \). Putting \( x = -x \) in (15), we have
\[
A(x)\beta y - A(y)\beta x + B(x,y)\beta x - B(x,y)\beta y = 0 \tag{16}
\]
Hence from (15) and (16) and by 2-torsion freeness, we obtain
\[
A(x)\beta y + B(x,y)\beta x = 0 \tag{17}
\]
Right multiplying (17) by \( \alpha A(x) \) and using (14) and (A), we get
\[
A(x)\alpha y \beta A(x) = 0
\]
Thus by semiprimeness of \( M \), we have \( A(x) = 0 \) for all \( x \in M \). i.e., \( T(x\alpha x) = T(x)\alpha x \). Therefore \( T \) is a left centralizer and hence \( T \) is a centralizer because of left-right symmetry.

It is obvious that if \( M \) is an arbitrary \( \Gamma \)-ring with involution and satisfying the assumption (A) and if \( T : M \to M \) is an additive mapping such that \( T(x\alpha I(x) + I(x)\alpha x) = T(x)\alpha I(x) + I(x)\alpha T(x) \) holds for any \( x \in M \) and \( \alpha \in \Gamma \), then \( T \) is a centralizer.

**Theorem 2.2** Let \( M \) be a 2-torsion free semiprime \( \Gamma \)-ring with \( I \)-involution and let \( T : M \to M \) be an additive mapping such that
\[
T(x\alpha I(x) + I(x)\alpha x) = T(x)\alpha I(x) + I(x)\alpha T(x) \tag{18}
\]
holds for \( x \in M \) and \( \alpha \in \Gamma \). If \( M \) has an identity element and \( M \) satisfy the assumption (A), then \( T(x) = a\alpha x \) for any \( x \in M \) and some \( a \in Z(M) \).
Proof. Putting $x = I(x) + e$, where $e$ denotes the identity element, in (18), after some calculation, we obtain

$$2T(x) = a\alpha x + x\alpha a$$

where $a = T(e)$. Replacing $x$ by $x\alpha I(x) + I(x)\alpha x$ in the above relation (19), we have

$$2T(x\alpha I(x) + I(x)\alpha x) = a\alpha(x\alpha I(x) + I(x)\alpha x) + (x\alpha I(x) + I(x)\alpha x)d(\alpha a)$$

Using (19) and (20) in (18) and using (A), we obtain

$$a\alpha I(x)\alpha x + I(x)\alpha x\alpha a = x\alpha a I(x) + I(x)\alpha a\alpha x$$

$$\Rightarrow [[a, I(x)]_\alpha, x]_\alpha = 0$$

$$\Rightarrow [d(I(x)), x]_\alpha = 0$$

(21)

where $d(I(x)) = [a, I(x)]_\alpha$. Linearization of the relation (21) for $x, y \in M$, we have

$$[d(I(x)), y]_\alpha + [d(I(y)), x]_\alpha = 0$$

(22)

Putting $x = I(x)$ in (22), we have

$$[d(x), y]_\alpha + [d(I(y)), I(x)]_\alpha = 0$$

(23)

Putting $y = I(a)$ in (23), we obtain

$$[d(x), I(a)]_\alpha = 0$$

(24)

as $d(x) = [a, x]_\alpha$. Putting $x = x\beta y$ in (24), we have

$$[d(x\beta y), I(a)]_\alpha = 0$$

$$\Rightarrow d(x)\beta [y, I(a)]_\alpha + [x, I(a)]_\alpha \beta d(y) = 0$$

(25)

Replacing $y$ by $z\gamma y$ in (25), we obtain

$$d(x)\beta z\gamma [y, I(a)]_\alpha + [x, I(a)]_\alpha \beta z\gamma d(y) = 0$$

(26)

Putting $x = I(a)$ and $y = a$ in (26), we have

$$d(I(a))\beta z\gamma d(I(a)) = 0$$
Since $z \in M$, hence by semiprimeness of $M$,

$$d(I(a)) = 0$$  \hspace{1cm} (27)

Replacing $y$ by $a$ in (23) and using (27), we have $[d(x), a]_{\alpha} = 0$, for all $x \in M$ and $\alpha \in \Gamma$, which implies $d^2(x) = 0$, for all $x \in M$. Hence from Lemma 2.3, $d = 0$ and hence $a \in Z(M)$. Since $M$ is 2-torsion free semiprime $\Gamma$-ring with involution, (19) gives $T(x) = a\alpha x$ for $x \in M$ and $\alpha \in \Gamma$.

**Conclusion**

In this study, we have given definition and example which have shown that $I$-involution exists on $\Gamma$-rings. We have proved that if $T$ is an Jordan centralizer on a 2-torsion free semiprime $\Gamma$-ring $M$ with $I$-involution, satisfying the assumption (A), then $T$ is a centralizer. Also we have shown that $T(x) = a\alpha x$ for any $x \in M$ and some $a \in Z(M)$, if $T$ is additive mapping such that $T(xaI(x) + I(x)a\alpha x) = T(x)aI(x) + I(x)aT(x)$ holds for $x \in M$, $\alpha \in \Gamma$ and if $M$ has an identity element and $M$ satisfy the assumption (A).

**References**


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