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Face-Distinguishing Facially-Proper Entire-Labeling of Plane Graphs

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Abstract

An entire-labeling (with positive integers) of a plane graph G is face-distinguishing, if $w(f_1) \neq w(f_2)$ for any two adjacent faces f_1 and f_2 of G, where $w(f)$ denote the sum of labels of the vertices and edges incident with f and also the label of f. In this paper we prove that every plane graph admits a face-distinguishing facially-proper entire-labeling with at most 12 labels. Moreover, we improve this bound for some classes of graphs.

Mathematics Subject Classification: 05C10, 05C78

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1 Introduction

In this paper we consider finite undirected graphs without loops and multiple edges.

A graph which can be embedded in the plane (i.e. it can be drawn in such a way that no edges cross each other) is called planar graph; a fixed embedding of a planar graph is called plane graph. Outerplane graphs are such plane graphs that every vertex lies on the outer face.

Let G be a plane graph. Let $V = V(G)$, $E = E(G)$ and $F = F(G)$ denote the vertex set, the edge set and the face set of G, respectively. Let

$c : V \cup E \cup F \to \{1, \ldots, k\}$ be an entire-labeling of G with positive integers $1, \ldots, k$. We write $v \in f$ ($e \in f$) if a vertex v (an edge e) is incident with a face f. For every face $f \in F$, we define its weight as

$$w(f) = c(f) + \sum_{v \in f} c(v) + \sum_{e \in f} c(e).$$

The smallest integer k for which G admits an entire-labeling with labels $1, \ldots, k$ such that $w(f_1) \neq w(f_2)$ for any two faces f_1 and f_2 of G is called entire face irregularity strength, which was introduced by Bača et al. [3]. A weaker condition is to require $w(f_1) \neq w(f_2)$ only when f_1 and f_2 are adjacent, see [5]. Note that such labelings need not be proper in a usual way (i.e. adjacent elements can receive the same label).

Two edges of a plane graph are face-adjacent if they are consecutive edges of a facial trail of some face. We say that an entire-labeling of a plane graph G is facially-proper if any two face-adjacent elements (two adjacent faces, two adjacent vertices, two face-adjacent edges, a vertex and an incident face, a vertex and an incident edge, an edge and an incident face) have different labels. Clearly, such a labeling is a relaxation of the classical proper entire-labeling (because two adjacent edges can receive the same label if they are not face-adjacent). We say that a facially-proper entire-labeling is face-distinguishing if $w(f_1) \neq w(f_2)$ for any two adjacent faces f_1 and f_2. This type of labeling is a relaxation of the labeling introduced by Bača et al. [3], because only adjacent faces must have different weights; on the other hand, it is a strengthening of the labeling defined in [5], since face-adjacent elements must receive different labels.

In this paper we deal with the following problems.

Question 1.1 *What is the minimum number of positive integers needed for a face-distinguishing facially-proper entire-labeling of a plane graph?*

Question 1.2 *What is the smallest integer k such that a plane graph admits a face-distinguishing facially-proper entire-labeling with labels from the set $\{1, \ldots, k\}$?*

2 Results

2.1 General upper bound

It is well-known that the vertices of every plane graph can be labeled with at most four labels such that adjacent vertices receive different labels.

Theorem 2.1 ([1]) *Every plane graph has a proper vertex-labeling with at most four labels.*

The dual G^* of a plane graph G can be obtained as follows: Corresponding to each face f of G there is a vertex f^* of G^*, and corresponding to each edge e of G there is an edge e^* of G^*; two vertices f^* and g^* are joined by the edge e^* in G^* if and only if their corresponding faces f and g are separated by the edge e in G (an edge separates the faces incident with it).

Corollary 2.2 *Every plane graph G has a proper face-labeling with at most four labels.*

Proof Let G^* be the dual of G. Clearly, G^* is also a plane graph. From Theorem 2.1 it follows that G^* has a proper vertex-labeling with at most four labels. This labeling induces a proper face-labeling of G in a natural way. $\square$

Two vertices u and v of a plane graph are 2-facially adjacent if there exists a facial segment of length (measured as its number of edges) at most two between them (i.e. they lie on the same face and are at distance at most two on that face). In a 2-facial vertex-labeling of a plane graph any two different 2-facially adjacent vertices receive distinct labels.

Theorem 2.3 ([6]) *Each plane graph has a 2-facial vertex-labeling using at most eight labels.*

A subdivision of a graph G is a graph resulting from a sequence of edge subdivisions applied to G. A subdivision of an edge $e = uv$ of the graph G is obtained by the addition of a new vertex w into G and replacement of uv with two new edges uw and wv.

Theorem 2.4 *Every plane graph G admits a face-distinguishing facially-proper entire-labeling with labels from the set $\{1, \ldots, 36\}$.*

Proof Let H be such a subdivision of G that every edge of G is subdivided exactly once. Clearly, H is a plane graph, hence it has a 2-facial vertex-labeling with at most eight labels, see Theorem 2.3. This labeling induces a total-labeling (a labeling of vertices and edges) of G in a natural way. Observe that in this labeling incident vertices and edges of G receive different labels, moreover, adjacent vertices, face-adjacent edges have different labels (because the corresponding vertices are 2-facially adjacent in H). Label the vertices and edges of G according to a 2-facial vertex-labeling of H with labels from the set $\{8, 12, 16, 20, 24, 28, 32, 36\}$.

From Corollary 2.2 it follows that G has a proper face-labeling with at most four labels. Label the faces of G with labels from the set $\{1, 2, 3, 4\}$.

Now we have a facially-proper entire-labeling c of G with 12 labels, moreover the maximum label is 36. Hence it is sufficient to show that this labeling is face-distinguishing.

Let f_1 and f_2 be adjacent faces of G. Observe that

$$\sum_{v \in f_1} c(v) + \sum_{e \in f_1} c(e) \equiv 0 \pmod 4,$$

$$\sum_{v \in f_2} c(v) + \sum_{e \in f_2} c(e) \equiv 0 \pmod 4.$$

Therefore, $w(f_1) \equiv c(f_1) \pmod 4$ and $w(f_2) \equiv c(f_2) \pmod 4$. Since the faces f_1 and f_2 are adjacent, they have different labels, i.e. $c(f_1) \neq c(f_2)$. Consequently, $w(f_1) \not\equiv w(f_2) \pmod 4$. This means that the faces f_1 and f_2 have different weights. □

Corollary 2.5 *Every plane graph has a face-distinguishing facially-proper entire-labeling with at most* 12 *labels.*

Proof It follows from the proof of Theorem 2.4. □

2.2 4-edge-connected plane graphs

The upper bound from Theorem 2.1 can be improved by one for triangle-free graphs.

Theorem 2.6 ([4]) *Every plane graph without triangles has a proper vertex-labeling with at most three labels.*

Theorem 2.7 *If G is a 4-edge-connected plane graph, then it admits a face-distinguishing facially-proper entire-labeling with labels from the set* $\{1, \ldots, 27\}$.

Proof Observe that each minimum edge-cut (edges whose removal disconnects G) of size g in G corresponds to a cycle of length g in G^* and vice versa, therefore, the edge connectivity of the graph G is equal to the girth (the length of a shortest cycle) of the dual graph G^*. The girth of G^* is at least four, since G is 4-edge-connected. Consequently, G^* is triangle-free. Theorem 2.6 implies, that G^* has a proper vertex-labeling with at most three labels. This labeling of G^* induces a proper face-labeling of G in a natural way. For a such labeling we use labels from the set $\{1, 2, 3\}$.

Next, similarly as in the proof of Theorem 2.4, label the vertices and edges of G according to a 2-facial vertex-labeling of the totally subdivided G with labels from the set $\{6, 9, 12, 15, 18, 21, 24, 27\}$.

Clearly, it holds $w(f_1) \not\equiv w(f_2) \pmod 3$ for any two adjacent faces f_1 and f_2 of G. □

Corollary 2.8 *If a plane graph has a proper face-labeling with at most three labels, then it admits a face-distinguishing facially-proper entire-labeling with at most* 11 *labels.*

2.3 Bipartite graphs

We remind the well-known characterization of bipartite graphs.

Lemma 2.9 *A graph is bipartite if and only if it has a proper vertex-labeling with at most two labels.*

The medial graph $M(G)$ of a plane graph G is obtained as follows. For each edge e of G insert a vertex $m(e)$ in $M(G)$. Join two vertices of $M(G)$ if the corresponding edges are face-adjacent (see [7], p. 47). Clearly, $M(G)$ is a plane graph, too.

Theorem 2.10 *Every bipartite plane graph G admits a face-distinguishing facially-proper entire-labeling with labels from the set $\{1, \ldots, 28\}$.*

Proof Let $M(G)$ be the medial of G. The graph $M(G)$ has a proper vertex-labeling with at most four labels, since it is a plane graph. Label the edges of G with labels $8, 12, 16, 20$ according to a proper vertex-labeling of $M(G)$. Since G is bipartite it has a proper vertex-labeling with labels $24, 28$ (see Theorem 2.9). Finally, label the faces with labels $1, 2, 3, 4$ such that adjacent faces receive different labels. To show that the obtained facially-proper entire-labeling is face-distinguishing we can use the same argument as in the proof of Theorem 2.4. □

Corollary 2.11 *Every bipartite plane graph admits a face-distinguishing facially-proper entire-labeling with at most* 10 *labels.*

2.4 Outerplane graphs

It is known that every outerplane graph contains a vertex of degree at most two, hence, the chromatic number of outerplane graphs is at most three.

Lemma 2.12 *If G is an outerplane graph, then it has a proper vertex-labeling with at most three labels.*

The weak dual of a plane graph G is the induced subgraph of the dual graph G^* whose vertices correspond to the bounded faces of G.

Theorem 2.13 ([2]) *The weak dual of any outerplane graph is a forest.*

Lemma 2.14 *If G is an outerplane graph, then it has a proper face-labeling with at most three labels.*

Proof From Theorem 2.13 it follows that the weak-dual of G is a forest. Clearly, every forest is a bipartite graph. Lemma 2.9 implies that every forest has a proper vertex-labeling with at most two labels. Such a labeling of the weak-dual induces a labeling of all faces except for the outerface of G with at most two labels. Adjacent faces of G receive distinct labels, because the corresponding vertices are adjacent in the weak-dual. If we label the outerface with the third label we obtain a proper face-labeling of G. □

Theorem 2.15 *Each outerplane graph G has a face-distinguishing facially-proper entire-labeling with labels from the set $\{1, \ldots, 24\}$.*

Proof The graph G has a proper face-labeling with labels $1, 2, 3$ (see Lemma 2.14) and a proper vertex-labeling with labels $6, 9, 12$ (see Lemma 2.12). Its medial graph $M(G)$ has a proper vertex-labeling with labels $15, 18, 21, 24$, which induces a facially-proper edge-labeling of G. Clearly, this entire-labeling of G is facially-proper, moreover, it holds $w(f_1) \not\equiv w(f_2) \pmod 3$ for any two adjacent faces f_1 and f_2 of G. □

Corollary 2.16 *Every outerplane graph has a face-distinguishing facially-proper entire-labeling with at most* 10 *labels.*

2.5 Plane graphs with bipartite duals

Theorem 2.17 *Let G be a plane graph such that its dual is bipartite. Then G admits a face-distinguishing facially-proper entire-labeling with labels from the set $\{1, \ldots, 18\}$.*

Proof Since G^* is bipartite it has a proper vertex-labeling with labels $1, 2$, which induces a proper face-labeling of G (with the same labels). Since G is a plane graph it has a proper vertex-labeling with at most four labels, say $4, 6, 8, 10$. The proper vertex-labeling of the medial graph $M(G)$ (with at most four labels) induces a facially-proper edge-labeling of G. For such a labeling we use labels $12, 14, 16, 18$. The obtained entire-labeling of G is facially-proper, moreover, it holds $w(f_1) \not\equiv w(f_2) \pmod 2$ for any two adjacent faces f_1 and f_2 of G. □

Corollary 2.18 *Let G be a plane graph such that its dual is bipartite. Then G has a face-distinguishing facially-proper entire-labeling with at most* 10 *labels.*

Theorem 2.19 *Let G be a bipartite plane graph such that its dual is also bipartite. Then G admits a face-distinguishing facially-proper entire-labeling with labels from the set $\{1, \ldots, 14\}$.*

Proof The graph G has a proper vertex-labeling with at most two labels, say $4, 6$, since it is bipartite, moreover, it has a proper face-labeling with at most two labels, say $1, 2$, since its dual is also bipartite. A proper vertex-labeling of the medial graph $M(G)$, with labels $8, 10, 12, 14$ induces a facially-proper edge-labeling of G. The obtained entire-labeling of G is facially-proper, moreover, it holds $w(f_1) \not\equiv w(f_2) \pmod 2$ for any two adjacent faces f_1 and f_2 of G. □

Corollary 2.20 *Let G be a bipartite plane graph such that its dual is also bipartite. Then G has a face-distinguishing facially-proper entire-labeling with at most 8 labels.*

2.6 Lower bound

Theorem 2.21 *There are infinitely many plane graphs such that any their face-distinguishing facially-proper entire-labeling uses at least 6 labels.*

Proof Any plane graph with at least two faces uses at least five labels in any face-distinguishing facially-proper entire-labeling, since if an edge is incident with two adjacent faces, then that edge, their endvertices and the incident faces must have different labels.

Let C_4 be a plane drawing of the cycle on four vertices. Assume that C_4 has a face-distinguishing facially-proper entire-labeling with five labels, say a, b, c, d, e. Assume that the faces of C_4 have labels a and b. Consequently, the vertices and edges are labeled with labels c, d, e. Let u and v be two vertices of C_4 which have the same label, say c. Assume that the label of their common neighbor w is d. Consequently, the adjacent edges uw and wv must have the same label e, a contradiction.

Let $H_1 = C_4$ and let H_2 be any plane graph. Let v_1 be a vertex of H_1 and let v_2 be a vertex of H_2 incident with the outerface. Let G be a plane graph obtained from H_1 and H_2 by identifying the vertices v_1 and v_2. Any face-distinguishing facially-proper entire-labeling of G uses at least six labels (we can use the same argument as above). □

3 Conclusion

We proved that every plane graph admits a face-distinguishing facially-proper entire-labeling with at most 12 labels. On the other hand, we showed that there are infinitely many plane graphs which require at least 6 labels for such a labeling. We improved the upper bound for some classes of graphs.

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