Even Sequential Harmonious Labeling On Path and Cycle Related Graphs

P. Selvaraju 1, P. Balaganesan 2,5, L. Vasu 3 and J. Renuka 4

1,2,3,4,5 Department of Mathematics
1 Vel Tech Multi Tech Dr. Rangarajan Dr. Sankanthula Engineering College
Chennai-600 062, India
2 Research Scholar, Hindustan University
Chennai-603 103, India
3 Easwari Engineering College
Chennai-600 089, India
4 Sri Sairam Engineering College
Chennai-600 044, India
5 Saveetha School of Engineering
Saveetha University
Chennai-602 105, Tamil Nadu, India

Copyright © 2014 P. Selvaraju et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

In this paper, we have shown that collection of paths $P_n^i$ where $n$ is odd, cycle, triangular snake, quadrilateral snake, $P_n^2$ are even sequential harmonious graph and also we prove that the cycle $C_n$ be a cycle $u_1, u_2, \cdots, u_n$ is an even sequential harmonious graph. Let $G$ be a graph with $V(G) = V(C_n) \cup \{w_i : 1 \leq i \leq n\}$, $E(G) = E(C_n) \cup \{u_iw_iu_{i-1}w_i : 1 \leq i \leq n\}$, then $G$ is an even sequential harmonious graph. Let $C_n$ be a cycle $u_1, u_2, \cdots, u_n$. Let $G$ be a graph with $V(G) = V(C_n)$ and $E(G) = E(C_n) \cup \{u_2u_n\}$, then $G$ is an even sequential harmonious graph and finally, we have shown that the crown $C_n \odot K_1$ is an even sequential harmonious graph.

Mathematics Subject classification: 05C78
Keywords: Path, cycle, triangular snake, quadrilateral snake, square graph, crown graph

1 Introduction

All graphs in this paper are finite, simple graphs and undirected graph. The symbols \( V(G) \) and \( E(G) \) denote the vertex set and edge set of a graph \( G \). The cardinality of the vertex set is called the order of \( G \). The cardinality of the edge set is called the size of \( G \). A graph with \( p \) vertices and \( q \) edges is called a \( G(p,q) \) graph [2].

A graph \( G = (V,E) \) with \( p \) vertices and \( q \) edges is said to be even sequential harmonious graph if it is possible to label the vertices \( x \in V \) with distinct elements \( F(x) \) from \( \{0, 1, 2, \cdots, 2q\} \) in such a way that when the edge \( e = uv \) is labeled with \( f(u) + f(v) \) if \( u + v \) is even and \( f(u) + f(v) + 1 \) if \( u + v \) is odd, then the resulting edge labels are distinct, \( f \) is called even sequential harmonious labeling of \( G \) [1].

2 Main Results

Theorem 2.1. Let \( \mathcal{A} \) be the collection of paths \( P_i^n \) where \( n \) is odd and \( P_i^n = U_i^n \cup U_{i+1}^{i+1} \cup \cdots \cup U_n^n \), \( 1 \leq i \leq m \). Let \( G \) be the graph obtained from \( \mathcal{A} \) with \( V(G) = \bigcup_{i=1}^n V(P_i^n) \) and \( E(G) = \bigcup_{i=1}^n E(P_i^n) \cup \left( U_{n+1}^{i+1} \cup U_{n+1}^{i+1} \right) 1 \leq i \leq m-1 \). Then \( G \) is an even sequential harmonious graph.

Proof. Define \( f : V(G) \to \{0, 1, 2, \cdots, 2q = 2(mn - 1)\} \) by
\[
\begin{align*}
f(u_i) & = i - 1, \\
f(u_{j}) & = f(u_{k-1}) + (j - 1), 2 \leq k \leq m, 1 \leq j \leq n.
\end{align*}
\]
Clearly, \( f \) is an even sequential harmonious labeling. \( \square \)

Theorem 2.2. Any cycle is an even sequential harmonious graph.

Proof. Let \( C_n \) be a cycle with vertices \( u_1, u_2 \cdots u_n \). Take \( n = \begin{cases} 2m, & \text{if } n \text{ is even;} \\ 2m + 1, & \text{if } n \text{ is odd}. \end{cases} \)

Define \( f : V(C_n) \to \{0, 1, 2, \cdots, 2(q = n)\} \) by
\[
\begin{align*}
f(u_i) & = 2i - 2, \quad 1 \leq i \leq m + 1. \\
f(u_{m+j}) & = \begin{cases} n - 2j + 3, & 2 \leq j \leq m, \text{if } n \text{ is even;} \\ n - 2j + 4, & 2 \leq j \leq (m + 1), \text{if } n \text{ is odd}. \end{cases}
\end{align*}
\]

The set of labels of the edges of \( C_n \) is \( \{2, 4, 6, \cdots, 2n\} \) and hence \( C_n \) is an even sequential harmonious graph. \( \square \)
**Theorem 2.3.** Any triangular snake is an even sequential harmonious graph.

*Proof.* Let $T_n$ be a triangular snake. Define $f : V(T_n) \rightarrow \{0, 1, 2, \cdots, 2(q = 3n - 3)\}$ by

\[
f(v_i) = 3i - 3, \quad 1 \leq i \leq n.
\]

\[
f(w_i) = 3i - 1, \quad 1 \leq i \leq n - 1.
\]

The label of the edge $v_{i-1}v_i$ is $6i - 8$, $2 \leq i \leq n$. The label of the edge $w_iv_i$ is $6i - 4$, $1 \leq i \leq n - 1$. The label of the edge $w_{i-1}v_i$ is $6i - 6$, $2 \leq i \leq n$. Hence $T_n$ is an even sequential harmonious graph.

**Theorem 2.4.** Any quadrilateral snake is an even sequential harmonious graph.

*Proof.* Let $Q_n$ denote a quadrilateral snake. Define $f : V(Q_n) \rightarrow \{0, 1, 2, \cdots, 2q\}$ by

\[
f(u_i) = 4i - 4, \quad 1 \leq i \leq n.
\]

\[
f(v_i) = 4i - 2, \quad 1 \leq i \leq n - 1.
\]

\[
f(w_i) = 4i - 1, \quad 1 \leq i \leq n - 1.
\]

The label of the edge $u_{i-1}v_i$ is $8i - 12$, $2 \leq i \leq n$. The label of the edge $u_iv_i$ is $8i - 6$, $1 \leq i \leq n - 1$. The label of the edge $u_{i+1}w_i$ is $8i$, $1 \leq i \leq n - 1$. The label of the edge $v_iw_i$ is $8i - 2$, $1 \leq i \leq n - 1$. Hence $Q_n$ is an even sequential harmonious graph.

**Theorem 2.5.** The graph $P_n^2$ is an even sequential harmonious graph.

*Proof.* Let $u_1u_2 \cdots u_n$ be the path $P_n$. Clearly $P_n^2$ has $n$ vertices and $2n - 3$ edges. Define $f : V(P_n^2) \rightarrow \{0, 1, 2, \cdots, 2q\}$ by

\[
f(u_i) = 2i - 2, \quad 1 \leq i \leq n - 1.
\]

\[
f(w_n) = 2n - 3.
\]

The label of the edge $u_1u_{i+1}$ is $4i - 2$, $1 \leq i \leq n - 1$. The label of the edge $u_iw_{i+1}$ is $4i$, $1 \leq i \leq n - 1$. Hence $P_n^2$ is an even sequential harmonious graph.

**Theorem 2.6.** Let $C_n$ be a cycle $u_1 u_2 \cdots u_n$. Let $G$ be a graph with $V(G) = V(C_n) \cup \{w_1; 1 \leq i \leq n\}$, $E(G) = E(C_n) \cup \{u_iw_i; u_{i+1}w_i; 1 \leq i \leq n\}$. Then $G$ is an even sequential harmonious graph.

*Proof.* Case(i): $n$ is odd.

Define $f : V(G) \rightarrow \{0, 1, 2, \cdots, 2q\}$ by

\[
f(u_i) = 3i - 3, \quad 1 \leq i \leq \frac{n-1}{2}.
\]

\[
f(w_i) = 3i - 1, \quad 1 \leq i \leq \frac{n-1}{2}.
\]

\[
f(u_{n+1}) = \frac{3n-1}{2}.
\]

\[
f(w_{n+1}) = \frac{3n+1}{2}.
\]

\[
f(u_{n+3}) = \frac{3n+9}{2}, \quad 1 \leq i \leq \frac{n-3}{2}.
\]
\[ f(\frac{w_{j+1}}{2}) = \frac{3n+5}{2}. \]
\[ f(\frac{w_{j+3}}{2}) = \frac{3n+7}{2}. \]
\[ f(\frac{w_{j+3}}{2}) = \frac{3n+7}{2+3n}, \] 1 \leq i \leq \frac{n-3}{2}.

Clearly, \( f \) is an even sequential harmonious labeling of \( G \).

**Case (ii):** \( n \) is even, \( n \geq 8 \).

Define \( f : V(G) \to \{0, 1, 2, \cdots, 2q = 6n\} \) by

\[ f(u_i) = 3. \]
\[ f(u_{i} - 4, 2 \leq i \leq \frac{n}{2}. \]
\[ f(u_{\frac{j}{2}+1}) = \frac{3n}{2} + 1. \]
\[ f(u_{\frac{j}{2}+i}) = \frac{3n}{2} - 2 + 3i, 2 \leq i \leq \frac{n-4}{2}. \]
\[ f(u_{n-1}) = 3n - 3. \]
\[ f(u_n) = 3n. \]
\[ f(w_1) = 0. \]
\[ f(w_2) = 7. \]
\[ f(w_1) = 3i + 1, 3 \leq i \leq \frac{n-2}{2}. \]
\[ f(w_2) = \frac{3n-2}{2}. \]
\[ f(w_{\frac{j}{2}+1}) = \frac{3n+12}{2}. \]
\[ f(w_{\frac{j}{2}+i}) = \frac{3n+12}{2} + 3i, 1 \leq i \leq \frac{n-8}{2}. \]
\[ f(w_{n-2}) = 3n - 5. \]
\[ f(w_{n-1}) = 3n - 4. \]
\[ f(w_n) = 3n - 1. \]

Clearly, \( f \) is an even sequential harmonious labeling of \( G \). \( \square \)

**Theorem 2.7.** Let \( C_n \) be a cycle \( u_1 u_2 \cdots u_n \). Let \( G \) be a graph with \( V(G) = V(C_n), E(G) = E(C_n) \cup \{u_2u_n\} \), then \( G \) is an even sequential harmonious graph.

**Proof.** Define \( f : V(G) \to \{0, 1, 2, \cdots, 2(q = n + 1)\} \) by

\[ f(u_1) = 0, f(u_n) = 2. \]
\[ f(u_i) = 2i, i = 2, 3, 4, \cdots, \frac{n}{2}, n \) is even. \]
\[ f(u_i) = 2i, i = 2, 3, 4, \cdots, \frac{n}{2}, n \) is odd. \]
\[ f(u_{\frac{j}{2}+j}) = n - 2j + 3, j = 1, 2, \cdots, \frac{n-2}{2}, n \) is even. \]
\[ f(u_{\frac{j+1}{2}+1}) = n - 2j + 3, j = 1, 2, \cdots, \frac{n-3}{2}, n \) is odd.

Clearly, \( f \) is an even sequential harmonious labeling of \( G \). \( \square \)

**Theorem 2.8.** The crown \( C_n \odot K_1 \) is an even sequential harmonious graph for all \( n \geq 3 \).

**Proof.** Let \( C_n \) be the cycle \( u_1, u_2, \cdots, u_n \) and let \( v_i \) be the vertex adjacent to \( u_i, 1 \leq i \leq n \).
Case(i): $n \equiv 0 \pmod{4}$
Define $f : V(C_n \odot K_1) \to \{0, 1, 2, \cdots, 2q\}$ by
\begin{align*}
f(u_{2i-1}) &= 4i - 3, \ 1 \leq i \leq \frac{n}{4}, \\
f(u_{2i}) &= 4i - 2, \ 1 \leq i \leq \frac{n}{4}, \\
f(u_{\frac{n+2}{2+2i-1}}) &= n + 4i, \ 1 \leq i \leq \frac{n}{4}.
\end{align*}
Clearly $f$ is an even sequential harmonious labeling.

Case(ii): $n \equiv 1 \pmod{4}$
Define $f : V(C_n \odot K_1) \to \{0, 1, 2, \cdots, q = 2n\}$ by
\begin{align*}
f(u_{2i-1}) &= 4i - 3, \ 1 \leq i \leq \frac{n-1}{4}, \\
f(u_{2i}) &= 4i - 2, \ 1 \leq i \leq \frac{n-1}{4}, \\
f(u_{\frac{n+1}{2+2i-1}}) &= n + 4i + 1, \ 1 \leq i \leq \frac{n+1}{4}.
\end{align*}
Clearly $f$ is an even sequential harmonious labeling.

Case(iii): $n \equiv 2 \pmod{4}$
Define $f : V(C_n \odot K_1) \to \{0, 1, 2, \cdots, q = 2n\}$ by
\begin{align*}
f(u_1) &= 1, \\
f(u_{2i}) &= 4i - 1, \ 1 \leq i \leq \frac{n-2}{4}, \\
f(u_{2i-1}) &= 4i - 4, \ 2 \leq i \leq \frac{n+2}{4}, \\
f(u_{\frac{n+2}{2+2i-1}}) &= n + 4i + 2, \ 1 \leq i \leq \frac{n-2}{4}.
\end{align*}
Clearly $f$ is an even sequential harmonious labeling.

Case(iv): $n \equiv 3 \pmod{4}$
Define $f : V(C_n \odot K_1) \to \{0, 1, 2, \cdots, q = 2n\}$ by
\begin{align*}
f(u_1) &= 1, f(u_{2i}) = 4i - 1, \ 1 \leq i \leq \frac{n-3}{4}, \\
f(u_{2i-1}) &= 4i - 4, \ 2 \leq i \leq \frac{n-3}{4}, \\
f(u_{\frac{n+3}{2+2i-1}}) &= n + 4i + 1, \ 0 \leq i \leq \frac{n-1}{4}.
\end{align*}
Clearly $f$ is an even sequential harmonious labeling.
\[ f(v_1) = 0. \]
\[ f(v_{2i}) = f(u_{2i}) - 1, \ 1 \leq i \leq \frac{n-3}{4}. \]
\[ f(v_{2i+1}) = f(u_{2i+1}) + 1, \ 1 \leq i \leq \frac{n+1}{4}. \]
\[ f(v_{\frac{2i+1}{2}}) = f(u_{\frac{2i+1}{2}}) - 2. \]
\[ f(v_{\frac{2i+1}{2}+1}) = f(u_{\frac{2i+1}{2}+1}) - 1, \ 1 \leq i \leq \frac{n-3}{4}. \]
\[ f(v_{\frac{n+1}{2}+2i}) = f(u_{\frac{n+1}{2}+2i}) - 1, \ 1 \leq i \leq \frac{n+1}{4}. \]

Clearly \( f \) is an even sequential harmonious labeling. \( \square \)

References


Received: June 7, 2014