Controllable Orbital Motion in a Neighborhood of Collinear Libration Point

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Abstract

In this work two stages of investigation of the controllable orbital movement of space vehicle in a neighborhood of the libration point $L_1$ of Earth-Sun system are considered. In the first stage, within the framework of the linearized Hill's problem, laws of control were built to compensate instability. At the same time control accelerations are directed along the Earth-Sun line. Then, the constructed control laws are investigated within the framework of a more adequate model of the motion - circular three-body problem. The results of the Numerical Modeling show the effectiveness of the proposed laws of control.

Keywords: restricted three-body problem, libration point, Hill's problem, Hamiltonian, Lyapunov stability, asymptotic stability, control, optimization

1 Introduction

In a during of long-term control of a orbital movement of a spacecraft or a space station in a neighborhood of the libration point $L_1$ or $L_2$ by Earth-
Sun system it’s desirable to construct rather simple laws of feedback control, ie, as a function of phase variables describing a position of the spacecraft. The basic principle for the construction of such control laws is the following. We consider a model of movement in a simplified setting, but preserving the main qualitative properties of the real movement. Under this formulation the control laws are designed in Hill’s problem and thereafter studied numerically for more adequate model - restricted three-body problem.

2 The equations of motion and the problem of stabilization.

The variant of equations of Hill’s model - special approximation of the restricted circular three-body problem [4] is taken as the mathematical model of motion. The equations of controllable orbital motion of space vehicle are presented in form

\[
\begin{align*}
\dot{x}_1 &= y_1 + x_2, & \dot{y}_1 &= -\frac{3x_1}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} + 2x_1 + y_2 + u; \\
\dot{x}_2 &= y_2 - x_1, & \dot{y}_2 &= -\frac{3x_2}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} - x_2 - y_1; \\
\dot{x}_3 &= y_3, & \dot{y}_3 &= -\frac{3x_3}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} - x_3,
\end{align*}
\]

where \(x = (x_1, x_2, x_3)\) – coordinates of position of space vehicle in rotate geocentric system of coordinates; \(y = (y_1, y_2, y_3)\) – impulses, \(u\) – control action directed along the line Earth-Sun.

The distance from the center of the Earth to the \(L_1\) (nearly 0.01 astronomical unit) is taken as the unit of distance, and time scales so that period of revolution around the Sun (one year) is equal to \(2\pi\) units of time. In this case, the velocity unit is equal to \(303,14\frac{11}{8}\), and the acceleration unit is equal to \(5,93 \cdot 10^{-5}\frac{m}{s^2}\).

Another variants of Hill’s problem equations closest to system (1) are published in [5], [1].

The system of equations of uncontrollable motion is obtained if in (1) we set \(u = 0\). This system is in Hamiltonian form with Hamiltonian

\[
H(x, y) = \frac{1}{2}||y||^2 - \frac{3}{||x||} - \frac{3}{2}x_1^2 + \frac{||x||^2}{2} + x_2y_1 - x_1y_2.
\]

In the rotate system the libration point \(L_1\) has coordinates

\[
x^* = (1, 0, 0), \quad y^* = (0, 1, 0).
\]

and it is the equilibrium point of uncontrollable system.
The qualitative properties of the orbital motion in neighborhoods of collinear libration points $L_1$ is natural to study with the help of the linearized system of equations, which has the form

$$
\begin{align*}
\dot{x}_1 &= x_2 + y_1, & \dot{y}_1 &= 8(x_1 - 1) + (y_2 - 1) + u(x, y); \\
\dot{x}_2 &= -x_1 + y_2, & \dot{y}_2 &= -4x_2 - y_1; \\
\dot{x}_3 &= y_3, & \dot{y}_3 &= -4x_3. \\
\end{align*}
$$

(4)

Uncontrollable system (4) with $u = 0$ has the set of eigenvalues

$$
\lambda_1 = \sqrt{1 + 2\sqrt{7}}, \lambda_2 = -\sqrt{1 + 2\sqrt{7}}, \lambda_3 = i\sqrt{2\sqrt{7} - 1}, \\
\lambda_4 = -i\sqrt{2\sqrt{7} - 1}, \lambda_5 = 2i, \lambda_6 = -2i.
$$

Since $\lambda_1$ is positive eigenvalue, the linear system (4) is unstable and there is a problem of the stabilization of motion.

During the research of properties investigation of the of controllability in the linear system (4) is easy to see that the system of the first four equations for the variables $(x_1, x_2, y_1, y_2)$, describing the motion in the ecliptic plane is completely controllable (Kalman criterion). At the same time, the system of six equations is singular - the last two equations for the variables $(x_3, y_3)$, describing the spatial motion, separated from the first four equations and control do not affect the changes in these variables. This leads to the fact that the qualitative properties of the controllable trajectories in the linear approximation, does not automatically carry over to the nonlinear case (system (1)) even near the collinear libration points $L_1$. Therefore, the proof of the stability of the stationary solution have to use the special properties of the system (1).

3 Stabilizing control.

For the construction of stabilizing control, which provides Lyapunov stability of the orbital motion in the neighborhood of $L_1$, it is possible to use the property of the Hamiltonian system (1), which has with $u = 0$. If we take the control as a function only of the variable $x_1$, then it is obvious that the system (1) is in Hamiltonian form. Consider the control of

$$
u^A_1(x_1) = a(x_1 - 1),$$

(5)

The Hamiltonian of system (1) with the control $u_1(x_1)$ is of the form

$$
H^*(x, y) = \frac{1}{2}||y||^2 - \frac{3}{2}x_1^2 + \frac{||x||^2}{2} + x_2y_1 - x_1y_2 - \frac{a}{2}(x_1 - 1)^2,
$$
and with $a < -9$ this Hamiltonian is the Lyapunov function in a neighborhood of the libration point [4]. Thus, control $u_1(x_1)$ is stabilizing and ensures Lyapunov stability of stationary solutions (3) of the system (1).

4 The construction of optimal controls.

Further studies will be linked with the concept of ”danger function” [2], [3]. This concept appears when considering the linear approximation of the control system (4). Eigenvalue $\lambda_1$ of system (4) is positive and implies that the instability of the stationary solution of (3) (collinear libration point $L_1$).

Denote by $d_1$ eigenvector row of the matrix $A$, i.e.

$$d_1A = \lambda_1d_1, \quad ||d_1|| = 1,$$

and form the linear function

$$l_1(z) \triangleq d_1z.$$

It is seen that the trajectories of the linearized system (4) holds

$$\dot{l}_1 = \lambda_1l_1$$

whence

$$l_1(t) = l_1(t_0) e^{\lambda_1(t-t_0)}$$

ie, the $l_1(t)$ increases exponentially. On the other hand, if the initial time danger function $l_1(x_0, y_0) = 0$, then setting $u = 0$, we obtain the linear approximation that the trajectory does not leave the neighborhood of libration points in time. Therefore, the stabilization of the orbital motion should be given priority to danger function.

We consider the mixed functional with danger function $l_1(x, y)$

$$J_1(u(\cdot)) = \int_{t_0}^{\infty} [k_1^2 l_1^2(x, y) + u^2] \, dt \to \min.$$  \hfill (6)

We pose the problem of constructing a controlled trajectory for the ”planar” variables of the system (4) - $(x_1, x_2, y_1, y_2)$, for which the functional $J_1$ reaches a minimum, ie linear-quadratic problem of optimization. The solution of this problem is the synthesis function $u_2 = u_2(x, y) -$ linear function of variables $(x, y)$ constructed by algebraic methods of linear-quadratic optimization.

To ensure the latter property, we consider the optimization of the functional

$$J_2(u(\cdot)) = \int_{t_0}^{\infty} [k_1^2 l_1^2(x, y) + k_0 \left( ||x||^2 + \alpha^2 ||y||^2 \right) + u^2] \, dt \to \min.$$  \hfill (7)
Solution to this problem is the synthesis function \( u_3 = u_3(x, y) \) – also linear function of variables \((x, y)\).

5 Numerical modeling.

In this research all control laws constructed in the framework of the Hill’s model. Numerical modeling is realized in the framework of the restricted three-body problem. The equations of uncontrollable movement in the framework of restricted three-body problem have the hamiltonian form with Hamiltonian

\[
H_1(x, y) = \frac{1}{2} ||y||^2 - \frac{3}{||x||} - R^2 \left( \frac{1}{\sqrt{1 - \frac{2x_1}{R}}} + ||x||^2 \right) + x_2y_1 - x_1y_2.
\]

The initial data for the trajectory and for all other trajectories of controllable motion, which will be illustrated below, are chosen the same, namely

\[
x_1(0) = 1.05, \ x_2(0) = 0, \ x_3(0) = 0.05, \ y_1(0) = 0, \ y_2(0) = 0.95, \ y_3(0) = 0. \tag{8}
\]

All graphs are constructed on the time interval of about two years. Figure 1 shows a graph of controllable motion with control \(u_1(x_1)\). The initial data of motion is presented in (8) and \( a = -10 \). Figure 2 shows the control as a function of time, i. e. \( u(t) = u_1(x(t)) \).

Figure 3 shows the behavior of the optimal trajectory with control \( u_2 \). Changing value of the control action over time is shown in Figure 4, which shows that the magnitude of control action decreases sharply at the beginning of the movement, and then changes its value in the range of the order of several hundredths of a unit of acceleration. The latter property is explained by the influence of nonlinearity, as control \( u_2 \) does not guarantee the asymptotic approximation to the libration point.

Figure 5 shows the plot of the space trajectory, and as shown in Figure 6 illustrates the behavior of the corresponding control \( u_3(x, y, \). It is seen that
in process of time the value of the control $u_3$ tends to a small constant, due to displacement of the libration point in the transition from the Hill’s model to circular three-body problem.

References


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