Existence and Stability of Periodic Solution
for an Immune System with Delays and Impulses

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Abstract

By using Mawhin continuation theorem and Lyapunov functional method, sufficient conditions ensuring the existence and stability of periodic solutions to an immune system with delays and impulses are established in this paper. The main results generalize and improve some known criteria [6].

Keywords Periodic solutions; Immune; Delays; Impulses

1 Introduction

In 1980, the following model of an infectious disease was proposed by G.I. Marchuk [1].

\[
\begin{aligned}
\dot{V}(t) &= (\beta - \gamma F(t))V(t) \\
\dot{C}(t) &= \alpha V(t - \tau)F(t - \tau) - \mu C(t) - C^* \ \\
\dot{F}(t) &= \rho C(t) - (\mu_F + \eta \gamma V(t))F(t)
\end{aligned}
\]

where \(\alpha\) is immune coefficient, which describes the state of infected organism. Coefficient \(\rho\) describes the rate of production of antibodies due to the presence of antigens. The basic properties of system (1) were studied by authors [2-5], but under seasonal changes of weather, the assumption that \(\alpha, \rho\) are all constants may not be fulfilled and such seasonal changes of the resistance may be observed in month’s cycle. Hence, authors [6] assumed coefficients \(\alpha(t), \rho(t)\) are positive continuous and bounded periodic functions.

1This paper is supported by National Natural Science Foundation of P.R. China (11161015,11161011) and Natural Science Foundation of Guangxi (2013GXNSFAA019003,2013GXNSFDA019001).
However, in real world, pulse vaccination is a usual strategy for people to stand against disease, which can cause the number of antigens and antibodies change abruptly. Hence, incorporating pulse vaccination and seasonality of changing environments, it is necessary and interesting for us to reconsider the existence and stability of system (1).

In this paper, we study the following immune system with delays and impulses:

\[
\begin{align*}
\dot{V}(t) &= (\beta - \gamma F(t))V(t) \\
\dot{C}(t) &= \alpha(t) V(t - \tau) F(t - \tau) - \mu_C(C(t) - C^*) \\
\dot{F}(t) &= \rho(t) C(t) - (\mu_F + \eta\gamma V(t))F(t) \\
V(t^+) &= (1 + p_k)V(t) \\
C(t^+) &= C(t) \\
F(t^+) &= F(t) + q_k
\end{align*}
\]

where \(V(t), C(t), F(t)\) denote antigen, plasmal cell and antibody concentration at time \(t\), respectively. \(\beta\) represents the production rate of antigens, \(\gamma\) is the coefficient expressing the probability of antigen-antibody encounters and their interaction, \(\alpha(t)\) represents the immune reactivity. Plasma cells’ production is delayed relative to B-cell stimulation process with a constant delay \(\tau\). If there is no antigens or antibodies, plasma cells’ concentration tends exponentially to physiological level \(C^*\) with coefficient \(\mu_C\), \(\mu_C^{-1}\) is the mean lifetime of a plasmal cell, \(\rho(t)\) describes the production rate of antibodies by plasma cells. \(\eta\) is the rate of antibodies necessary to suppress one antigen, \(\mu_F^{-1}\) is the mean antibody lifetime. \(p_k\) is the proportion of those vaccinated successfully and \(q_k\) is the dose by injection at time \(t = t_k\). Without loss of generality, we assume that \([0, \omega] \cap \{t_k\} = \{t_1, t_2, \cdots, t_m\}\). For more details, we may refer to [1-8].

System (2) is accompanied with the following initial conditions

\[(V(s), C(s), F(s)) = (V_0(s), C_0(s), F_0(s)), s \in [-\tau, 0]\]

where \(V_0, C_0, F_0\) are continuous non-negative functions on \([-\tau, 0]\).

From the biological point of view, we only consider (2) in the biological meaning region \(D = \{(V, C, F) | V \geq 0, C \geq 0, F \geq 0\}\).

By using Mawhin continuation theorem and functional method, we aim to establish sufficient conditions of the existence and globally asymptotical stability to periodic solutions of (2).

Throughout this paper, we assume that
(i) \(0 \leq t_1 < t_2 < \cdots < t_n < \cdots\) and \(\lim_{k \to \infty} t_k = \infty\);
(ii) \(\beta, \gamma, \tau, \mu_C, \mu_F, C^*, \eta, 0 \leq p_k < 1, 0 \leq q_k < 1\) are all positive constants;
(iii) \(\alpha(t), \rho(t)\) are positive \(\omega\)-periodic, continuously differentiable and uniformly bounded functions;
(iv) There exists a constant \(m \in \mathbb{Z}^+\) such that \(t_{k+m} = t_k + \omega, p_{k+m} = p_k, q_{k+m} = q_k\).
For convenience, we use the following notations.

\[ PC = \{ x \in R : R \rightarrow R^+, \text{lim}_{s \rightarrow t} x(s) = x(t) \text{ if } t \neq t_k; \text{lim}_{t \rightarrow t_k^+} x(t) = x(t_k), \text{lim}_{t \rightarrow t_k^-} x(t) \text{ exists, } k \in Z^+ \}; \]

\[ PC_{\omega} = \{ x \in PC, x(t + \omega) = x(t) \}, PC'_{\omega} = \{ x \in PC', x(t + \omega) = x(t) \}; \]

\[ f^M = \max_{t \in [0, \omega]} f(t), \quad f^m = \min_{t \in [0, \omega]} f(t), \text{ where } f \text{ is a positive } \omega \text{-periodic function.} \]

The rest of this paper is organized as follows. In Section 2, some preliminaries are introduced. In Section 3, the existence and globally asymptotical stability of periodic solutions of (2) are studied. In section 4, an illustrative example is given to show the effectiveness of the main results.

2 Preliminaries

First, we make some preparations. Let \( X, Z \) be real Banach spaces, \( L : \text{Dom}L \subset X \rightarrow Z \) be a linear mapping, and \( N : X \rightarrow Z \) be a continuous mapping. The mapping \( L \) is called a Fredholm mapping of index zero if \( \text{dimKer}L = \text{codimIm}L < +\infty \) and \( \text{Im}L \) is closed in \( Z \). If \( L \) is a Fredholm mapping of index zero and there exist continuous projectors \( L_L : X \rightarrow X \) and \( Q : Z \rightarrow Z \) such that \( \text{Im}P = \text{Ker}L, \text{Ker}Q = \text{Im}L = \text{Im}(I - Q) \), then the restriction \( L_p \) of \( L \) on \( \text{Dom}L \cap \text{Ker}P : (I - P)X \rightarrow \text{Im}L \) is invertible. Denote the inverse of \( L_p \) by \( K_p \). If \( \Omega \) is an open bounded subset of \( X \), the mapping \( N \) will be called \( L \)-compact on \( \Omega \) if \( N(\Omega) \) is bounded and \( K_p(I - Q)N : \Omega \rightarrow X \) is compact. Since \( \text{Im}Q \) is isomorphic to \( \text{Ker}L \), there exists isomorphism \( J : \text{Im}Q \rightarrow \text{Ker}L \).

**Definition 1** Functions \( (V(t), C(t), F(t))(\in C([-\tau, \infty), R^+)) \) is said to be a solution of (2) on \( [0, \omega] \) if

(i) \( V(t), C(t), F(t) \) are absolutely continuous on each interval \( (0, t_1) \) and \( (t_k, t_{k+1}) \);

(ii) for any \( t_k \), \( V(t_k^+), C(t_k^+), F(t_k^+) \) exist and \( V(t_k^-) = V(t_k), C(t_k^-) = C(t_k), F(t_k^-) = F(t_k) \);

(iii) \( V(t), C(t), F(t) \) satisfy (2) for almost everywhere in \( [0, \omega] \setminus \{ t_k \} \) and satisfy \( V(t_k^+) - V(t_k) = \rho_{\alpha} V(t_k), C(t_k^+) = C(t_k), F(t_k^+) - F(t_k) = q_k \) for every \( t = t_k \).

**Definition 2** Set \( A \subset PC_{\omega} \) is said to be quasi-equicontinuous in \([0, \omega]\) if for any \( \varepsilon > 0 \), there exists \( \delta > 0 \), if \( x \in A, k \in N, t', t'' \in (t_{k-1}, t_k) \cap [0, \omega] \), and \( |t' - t''| < \delta \), then \( |x(t') - x(t'')| < \varepsilon \).

**Lemma 1**[9] Set \( A \subset PC_{\omega} \) is relatively compact if and only if:

(1) \( A \) is bounded, that is, \( \|x\| \leq c \) for each \( x \in A \) and some \( c > 0 \);

(2) \( A \) is quasi-equicontinuous in \([0, \omega]\).

**Lemma 2**[10] (Mawhin continuation theorem) Let \( \Omega \subset X \) be an open bounded set, \( L \) be a Fredholm mapping of index zero and \( N \) be \( L \)-compact on \( \Omega \). Further,

(a) For each \( \lambda \in [0, 1], x \in \partial \Omega \cap \text{Dom}L, Lx \neq \lambda Nx; \)

(b) For each \( x \in \partial \Omega \cap \text{Ker}L, QNx \neq 0; \)

(c) \( \text{deg}\{JQN, \Omega \cap \text{Ker}L, 0\} \neq 0; \)

Then \( Lx = Nx \) has at least one solution in \( \overline{\Omega} \cap \text{Dom}L \).

**Definition 3** A bounded positive solution \((\bar{V}(t), \bar{C}(t), \bar{F}(t))\) of (2) is said to be globally asymptotically stable if for any other positive bounded solution \((V(t), C(t), F(t))\) of (2), the following
equality holds:
$$\lim_{t \to \infty} (|V(t) - \tilde{V}(t)| + |C(t) - \tilde{C}(t)| + |F(t) - \tilde{F}(t)|) = 0.$$  

**Lemma 3** Let $h$ be a real number and $f$ be non-negative, integrable and uniformly continuous on $[h, \infty)$, then $\lim_{t \to \infty} f(t) = 0$.

### 3 Main results

In this section, we study the existence and globally asymptotical stability of positive $\omega$-periodic solution of (2).

**Theorem 1** Assume that

(C1) $\tau > 0$ is small enough, or $\tau \to 0$,

(C2) $\rho^m C^\ast \omega + \sum_{k=1}^{m} q_k > \mu_F \sigma_1 + \eta \sum_{k=1}^{m} \ln(1 + p_k)$,

(C3) $\sigma_1 \mu_F \sigma_1 > \sigma_2 \sum_{k=1}^{m} q_k + \eta \gamma C^\ast \omega$,

(C4) $\rho^M \sigma_2 - \mu \sigma_1 + \sum_{k=1}^{m} q_k - \eta \sum_{k=1}^{m} \ln(1 + p_k) > 0$.

where $\sigma_1, \sigma_2$ are defined in (6) and (17) respectively, then (2) has at least one positive $\omega$-periodic solution.

**Proof.** Make the change of variable $V(t) = e^{W(t)}$, then system (2) is reformulated as:

$$\begin{align*}
\dot{W}(t) &= \beta - \gamma F(t), \\
\dot{C}(t) &= \alpha(t)e^{W(t-\tau)} F(t-\tau) - \mu_C(C(t) - C^\ast), \\
\dot{F}(t) &= \rho(t)C(t) - (\mu_F + \eta \gamma e^{W(t)}) F(t), \\
W(t^+) &= W(t) + \ln(1 + p_k), \\
C(t^+) &= C(t), \\
F(t^+) &= F(t) + q_k,
\end{align*}$$

(3)

To complete the proof, it suffices to show (3) has at least one positive $\omega$-periodic solution.

In order to use Lemma 2 to (3), let $X = \{x = (W(t), C(t), F(t))^T | W(t), C(t), F(t) \in PC_\omega, Z = X \times \mathbb{R}^{3m} \}$ with norms $\|x\|_X = \sup |W(t)| + \sup |C(t)| + \sup |F(t)|, \|z\|_Z = \|x\|_X + \|y\|, z = (x, y) \in Z$, respectively, where $\|\cdot\|$ is the Euclidean norm of $\mathbb{R}^{3m}$, then $X, Z$ are all Banach spaces.

Define $L : \text{Dom}L \subset X \to Z, Lx = (x, \Delta x(t_1), \Delta x(t_2), \cdots, \Delta x(t_m))$, $N : X \to Z, Nx = \begin{bmatrix} \beta - \gamma F(t) \\ \alpha(t)e^{W(t-\tau)} F(t-\tau) - \mu_C(C(t) - C^\ast) \\ \rho(t)C(t) - (\mu_F + \eta \gamma e^{W(t)}) F(t) \end{bmatrix} \begin{bmatrix} \ln(1 + p_k) \\ 0 \\ q_k \end{bmatrix}_{k=1}^{m}$,

where $\text{Dom}L = \{x = (W(t), C(t), F(t))^T | W(t), C(t), F(t) \in PC_\omega, Z$. It is not difficult to verify that $\text{Ker}L = \{x \in X | x = h \in \mathbb{R}^3\}$, $\text{Im}L = \{z = (f_1, c_1, \cdots, c_m) \in Z \bigg| \int_0^{\omega} f(s) ds + \sum_{k=1}^{m} c_k = 0 \}$, and $\dim \text{Ker}L = \text{codim} \text{Im}L = 3$. Thus, $\text{Im}L$ is closed in $Z$ and $L$ is a Fredholm mapping of index zero.
Take $P_\omega = \frac{1}{\omega} \int_0^\omega x(t)dt$, $x \in X, Q_\omega = Q(f, b_1, \cdots, b_m) = \left( \frac{1}{\omega} \left[ \int_0^\omega f(s)ds + \sum_{k=1}^m b_k \right] , (0, \cdots, 0)_{3 \times m} \right)$.

Evidently, $P_\omega$ and $Q_\omega$ are continuous such that \( \text{Im} P = \text{Ker} L, \text{Im} Q = \text{Im} (I - Q) \). Hence, the general inverse (to $L$) $K_p : \text{Im} L \to \text{Ker} p \cap \text{Dom} L$ has the form:

$$K_p(z) = \int_0^t f(s)ds + \sum_{t>t_k} b_k - \frac{1}{\omega} \int_0^t \int_0^{t_k} f(s)dsdt - \sum_{k=1}^m b_k + \frac{1}{\omega} \sum_{k=1}^m b_k t_k.$$

Therefore,

$$QN_x = \left( \left( \frac{1}{\omega} \int_0^\omega (\beta - \gamma F(t)) dt + \frac{1}{\omega} \sum_{k=1}^m \ln(1 + p_k) \right) , \left\{ \left. \int_0^t (\beta - \gamma F(t)) dt + \sum_{t>t_k} \ln(1 + p_k) \right\} \right) , \left\{ \begin{array}{l} 0 \\ 0 \\ \vdots \\ 0 \end{array} \right\}_{k=1}^{m}.$$

$$K_p(I - Q)N_x = \left( \left( \frac{1}{\omega} \int_0^\omega (\beta - \gamma F(t)) dvdt + \sum_{t>t_k} \ln(1 + p_k) - \frac{1}{\omega} \sum_{k=1}^m \ln(1 + p_k) t_k \right) , \left\{ \left. \int_0^t (\beta - \gamma F(t)) dvdt + \sum_{t>t_k} \ln(1 + p_k) - \frac{1}{\omega} \sum_{k=1}^m \ln(1 + p_k) t_k \right\} \right) , \left\{ \begin{array}{l} 0 \\ 0 \\ \vdots \\ 0 \end{array} \right\}_{k=1}^{m}.$$

It is not difficult to show that $QN$ and $K_p(I - Q)N$ are continuous. By applying Ascoli-Arzelà theorem, moreover, \( QN(\Omega), K_p(I - Q)N(\Omega) \) are relatively compact for any open bounded set \( \Omega \subset X \). Hence, $N$ is $L$-compact on $\Omega$ for any open bounded set $\Omega$ in $X$.

Now we reach the position to search for an appropriate open bounded subset $\Omega$ for the application of Lemma 2. Corresponding to equation $Lx = \lambda N x, \lambda \in (0, 1)$, we have

$$
\begin{align*}
\hat{W}(t) &= \lambda(\beta - \gamma F(t)) \\
\hat{C}(t) &= \lambda(\alpha(t)e^{\omega(t-\tau)}F(t-\tau) - \mu C(C(t) - C^*)) \\
\hat{F}(t) &= \lambda(\rho(t)C(t) - (\mu_F + \eta \gamma e^{\omega(t)})F(t) \\
W(t_k^+) - W(t_k) &= \lambda \ln(1 + p_k) \\
C(t_k^+) - C(t_k) &= 0 \\
F(t_k^+) - F(t_k) &= \lambda q_k
\end{align*}
$$

(4)

Let \( \hat{x}(t) = (\hat{W}(t), \hat{C}(t), \hat{F}(t))^T \) be an $\omega$-periodic solution of (4) for some certain $\lambda \in (0, 1)$. Integrating the first and fourth equations of (4) over the interval $[0, \omega]$, we have

$$\beta \omega + \sum_{k=1}^m \ln(1 + p_k) = \gamma \int_0^\omega F(t)dt.$$

(5)

Then

$$\int_0^\omega F(t)dt = \frac{\beta \omega + \sum_{k=1}^m \ln(1 + p_k)}{\gamma} = \sigma_1 > 0.$$

(6)
By the continuity of $F(t)$ on the interval $[0, \omega]$, there exist $\xi_1, \eta_1 \in [0, \omega]$ such that

$$F(\xi_1) = \min_{t \in [0, \omega]} F(t), \quad F(\eta_1) = \max_{t \in [0, \omega]} F(t).$$

Then

$$F(\xi_1) \leq \frac{\sigma_1}{\omega}, \quad F(\eta_1) \geq \frac{\sigma_1}{\omega}.$$  \hfill (7)

Integrating the second equation of system (3) on interval $[0, \omega]$, we have

$$\mu_C \int_0^\omega C(t) dt \geq C^* \mu_C \omega.$$  

Let $C(\eta_2) = \max_{t \in [0, \omega]} C(t)$, then

$$C(\eta_2) \geq C^*.$$  \hfill (8)

Integrating the third and sixth equations of (3) on $[0, \omega]$ leads to

$$\eta \gamma \int_0^\omega e^{W(t)} F(t) dt = \int_0^\omega \rho(t) C(t) dt - \mu_F \int_0^\omega F(t) dt + \sum_{k=1}^m q_k \geq \rho^m C^* \omega - \mu_F \sigma_1 + \sum_{k=1}^m q_k.$$  \hfill (9)

On the other hand, by the first equation of (2), we have

$$\gamma \int_0^\omega F(t) V(t) dt = \beta \int_0^\omega V(t) dt + \sum_{k=1}^m \ln(1 + p_k).$$  \hfill (10)

Substituting $V(t) = e^{W(t)}$ into (10), then

$$\gamma \int_0^\omega F(t) e^{W(t)} dt = \beta \int_0^\omega e^{W(t)} dt + \sum_{k=1}^m \ln(1 + p_k).$$  \hfill (11)

It follows from (9) and (11) that

$$\int_0^\omega e^{W(t)} dt \geq \frac{\rho^m C^* \omega - \mu_F \sigma_1 + \sum_{k=1}^m q_k - \eta \sum_{k=1}^m \ln(1 + p_k)}{\eta \beta}. \hfill (12)$$

By (12) and the continuity of $W(t)$, then there exists $\eta_3 \in [0, \omega]$ such that $W(\eta_3) = \max_{t \in [0, \omega]} W(t)$ and

$$W(\eta_3) \geq \ln \frac{\rho^m C^* \omega - \mu_F \sigma_1 + \sum_{k=1}^m q_k - \eta \sum_{k=1}^m \ln(1 + p_k)}{\eta \beta \omega} = m_0.$$  \hfill (13)

In view of (C1) and (3), we deduce that

$$0 \geq \alpha^m \int_0^\omega e^{W(t - \tau)} F(t - \tau) dt - \mu_C \int_0^\omega C(t) dt + \mu_C C^* \omega$$

$$- \alpha^m \int_0^\omega e^{W(t - \tau)} F(t - \tau) dt - \mu_C \int_0^\omega C(t) dt + \mu_C C^* \omega,$$ \hfill (14)

$$0 \geq \sum_{k=1}^m q_k + \rho^m \int_0^\omega C(t) dt - \mu_F \int_0^\omega F(t) dt - \eta \gamma \int_0^\omega e^{W(t)} F(t) dt.$$ \hfill (15)

$\eta \gamma \times (14) + \alpha^m \times (15)$ leads to

$$0 \geq \alpha^m \sum_{k=1}^m q_k + (\rho^m \alpha^m - \eta \gamma \mu_C) \int_0^\omega C(t) dt + \eta \gamma \mu_C C^* \omega - \mu_F \alpha^m \sigma_1.$$  \hfill (16)
By (C2), we deduce that \( \rho^m C^* \omega - \mu_F \sigma_1 + \sum_{k=1}^{m} q_k > 0 \). Combining (C3), we have \( \alpha^m \rho^m C^* \omega > \alpha^m \mu_F \sigma_1 - \alpha^m \sum_{k=1}^{m} q_k > \eta \gamma \mu_C C^* \omega \). That is, \( \alpha^m \rho^m > \eta \gamma \mu_C \). Hence, by (16), we have

\[
\int_{0}^{\omega} C(t) dt \leq \int_{0}^{\omega} F(t) dt - \eta \int_{0}^{\omega} e^{W(t)} F(t) dt.
\]

By the third equation of (3), then

\[
- \sum_{k=1}^{m} q_k = \int_{0}^{\omega} \rho(t) C(t) dt - \mu_F \int_{0}^{\omega} F(t) dt - \eta \int_{0}^{\omega} e^{W(t)} F(t) dt.
\]

By (C4), (11) and (18), then \( \eta \left( \beta \int_{0}^{\omega} e^{W(t)} dt + \sum_{k=1}^{m} \ln(1 + p_k) \right) \leq \rho^M_2 - \mu_F \sigma_1 + \sum_{k=1}^{m} q_k \), i.e.,

\[
\int_{0}^{\omega} e^{W(t)} dt \leq \frac{\rho^M \sigma_2 - \mu_F \sigma_1 + \sum_{k=1}^{m} q_k - \eta \sum_{k=1}^{m} \ln(1 + p_k)}{\eta \beta} \leq 3.
\]

By the continuity of \( C(t) \) and \( F(t) \), there exist \( \xi_2, \xi_3 \) such that

\[
C(\xi_2) = \min_{t \in [0, \omega]} C(t), \quad W(\xi_3) = \min_{t \in [0, \omega]} W(t).
\]

From (17), (19) and (20), we have

\[
C(\xi_2) \leq \frac{\sigma_2}{\omega}, \quad W(\xi_3) \leq \ln \frac{\sigma_3}{\omega}.
\]

On the other hand, we can obtain from (3) that:

\[
\int_{0}^{\omega} \left| \dot{W}(t) \right| dt \leq 2 \beta \omega + \sum_{k=1}^{m} \ln(1 + p_k), \quad \int_{0}^{\omega} \left| \dot{C}(t) \right| dt \leq 2(\rho^m \sigma_2 + \mu_F \sigma_1) + \sum_{k=1}^{m} q_k.
\]

Therefore, we have

\[
W(t) \leq W(\xi_3) + \int_{0}^{\omega} \left| \dot{W}(t) \right| dt \leq \ln \frac{\sigma_3}{\omega} + 2 \beta \omega + \sum_{k=1}^{m} \ln(1 + p_k) = M_1.
\]

\[
C(t) \leq C(\xi_2) + \int_{0}^{\omega} \left| \dot{C}(t) \right| dt \leq \frac{\sigma_2}{\omega} + 2 \mu_C C^* \omega + \mu_C \sigma_2 = M_2,
\]

\[
F(t) \leq F(\xi_1) + \int_{0}^{\omega} \left| \dot{F}(t) \right| dt \leq \frac{\sigma_1}{\omega} + 2(\rho^m \sigma_2 + \mu_F \sigma_1) + \sum_{k=1}^{m} q_k = M_3.
\]

Similarly we can derive that

\[
W(t) \geq W(\eta_3) - \int_{0}^{\omega} \left| \dot{W}(t) \right| dt \geq m_0 - 2 \beta \omega - \sum_{k=1}^{m} \ln(1 + p_k) = m_1.
\]

\[
C(t) \geq C(\eta_2) - \int_{0}^{\omega} \left| \dot{C}(t) \right| dt \geq C^* - 2 \mu_C C^* \omega - \mu_C \sigma_2 = m_2,
\]

\[
F(t) \geq F(\eta_1) - \int_{0}^{\omega} \left| \dot{F}(t) \right| dt \geq \frac{\sigma_1}{\omega} - 2(\rho^m \sigma_2 + \mu_F \sigma_1) - \sum_{k=1}^{m} q_k = m_3.
\]
Clearly, \( m_i, M_i (i = 1, 2, 3) \) are independent of \( \lambda \). Take \( H_i = \max\{|m_i|, |M_i|\} \) and \( H = 3 \max H_i + H_0 \) for \( i = 1, 2, 3 \), where \( H_0 \) is taken sufficiently large such that the unique solution \((\tilde{W}, \tilde{C}, \tilde{F})\) of the following system

\[
\begin{cases}
\int_0^\omega (\beta - \gamma F) dt + \sum_{k=1}^m \ln(1 + p_k) = 0, \\
\int_0^\omega (\alpha(t) e^W F - \mu_C (C - C^*) ) dt = 0, \\
\int_0^\omega (\rho(t) C - (\mu_F + \eta \gamma e^W F)) dt + \sum_{k=1}^m q_k = 0,
\end{cases}
\]

satisfies \( \| (\tilde{W}, \tilde{C}, \tilde{F}) \| = |\tilde{W}| + |\tilde{C}| + |\tilde{F}| < H_0 \). Take \( \Omega = \{ x = (W(t), C(t), F(t)) \| x \| < H \} \), then \( \Omega \) is bounded open subset of \( X \). With the help of (22-27), it is easy to follow that \( \Omega \) satisfies the requirement (a) of Lemma 2. Moreover, \( Q Nx \neq 0 \) for \( x \in \partial \Omega \cap R^3 \). A direct computation gives \( \deg \{ JQN x, \Omega \cap \ker L, 0 \} \neq 0 \). Here \( J \) is taken as the identity mapping since \( \text{Im} Q = \ker L \).

So far we have proved that \( \Omega \) satisfies all the requirements of Lemma 2. Therefore, by Lemma 2, (3) has at least one positive \( \omega \)-periodic solutions, which leads to that (2) has at least one \( \omega \)-periodic solutions. This completes the proof.

**Remark 1** The method used in the proof of Theorem 1 is much different from those employed in reference [6]. Particularly, conditions here are more easily verified than those of [6]. If \( p_k = q_k = 0 \), we can obtain the existence of periodic solution of (2) without impulse. Hence we generalize and improve the results of [6].

**Theorem 2** Assume (C1) - (C4) hold. Furthermore,

\[
\begin{align*}
(C_5) & \quad \text{sgn}(C(t) - \tilde{C}(t)) = -\text{sgn}(e^{W(t)} F(t) - e^{W(t)} \tilde{F}(t)); \\
(C_6) & \quad \gamma - \mu_F < 0, \eta \gamma - \alpha M < 0, \rho^M - \mu_C < 0.
\end{align*}
\]

Then the positive periodic solutions of (2) are globally asymptotically stable.

**Proof.** Define an functional as \( H(t) = |W(t) - \tilde{W}(t)| + |C(t) - \tilde{C}(t)| + |F(t) - \tilde{F}(t)| \). Calculating the derivative of \( H(t) \) at \( t = t_k \) along solutions of (3), combining (C5), then

\[
\dot{H}(t) = \text{sgn}(W(t) - \tilde{W}(t))(-\gamma F(t) + \gamma \tilde{F}(t)) + \text{sgn}(C(t) - \tilde{C}(t))(\alpha(t)e^{W(t)} F(t) - \mu_C C(t)) \\
- \alpha(t)e^{W(t)} \tilde{F}(t) + \mu_C \tilde{C}(t) + \text{sgn}(F(t) - \tilde{F}(t))(\rho(t) C(t) - \mu_F F(t) - \eta \gamma e^{W(t)} F(t)) \\
- \rho(t) \tilde{C}(t) + \mu_F \tilde{F}(t) + \eta \gamma e^{W(t)} \tilde{F}(t)) \\
\leq (\gamma - \mu_F)|F(t) - \tilde{F}(t)| + (\eta \gamma - \alpha M)|e^{W(t)} F(t) - e^{W(t)} \tilde{F}(t)| + (\rho^M - \mu_C)|C(t) - \tilde{C}(t)|
\]

In virtue of (H6), we have

\[
H(t) + \int_T^\infty (\mu_F - \gamma)|F(t) - \tilde{F}(t)| + (\alpha M - \eta \gamma)|e^{W(t)} F(t) - e^{W(t)} \tilde{F}(t)| + (\mu_C - \rho^M)|C(t) - \tilde{C}(t)| \leq H(T) < \infty.
\] (28)

If \( t = t_k \), then it is easy to show that

\[
H(t_k^+) \leq H(t_k).
\] (29)

By (28) and (29), we have \( |F(t) - \tilde{F}(t)| + |e^{W(t)} F(t) - e^{W(t)} \tilde{F}(t)| + |C(t) - \tilde{C}(t)| \in L^1(T, \infty) \).

In addition, from the boundedness of \( W(t), C(t) \) and \( F(t) \), we obtain that the derivatives of \( W(t), \tilde{W}(t), C(t) \) and \( F(t), \tilde{F}(t) \) are all bounded for all \( t \geq T \). Therefore, \( |F(t) - \tilde{F}(t)| + |e^{W(t)} F(t) - e^{W(t)} \tilde{F}(t)| + |C(t) - \tilde{C}(t)| \) is uniformly continuous. By Lemma 3, then

\[
\lim_{t \to \infty} |W(t) - \tilde{W}(t)| + |C(t) - \tilde{C}(t)| + |F(t) - \tilde{F}(t)| = 0.
\] (30)
(30) leads to \( \lim_{t \to \infty} |W(t) - \tilde{W}(t)| = 0, \lim_{t \to \infty} |C(t) - \tilde{C}(t)| = 0, \lim_{t \to \infty} |F(t) - \tilde{F}(t)| = 0. \) That is, positive periodic solutions of (3) are globally asymptotically stable. Therefore, positive periodic solutions of (2) are globally asymptotically stable. This completes the proof.

4 An illustrative example

In this section, we give an example to show the effectiveness of the main results.

Example. Consider the following system:

\[
\begin{align*}
\dot{V}(t) &= \left( \frac{1}{12} - \frac{1}{36} F(t) \right)V(t), \\
\dot{C}(t) &= \left( \frac{1}{3} + \frac{1}{3} \sin 2\pi t \right)V(t - \tau)F(t - \tau) - \left( \frac{1}{4} + \frac{1}{12} V(t) \right)F(t), \\
\dot{F}(t) &= \left( \frac{1}{3} + \frac{1}{6} \cos 2\pi t \right)C(t) - \left( \frac{1}{4} + \frac{1}{36} V(t) \right)F(t), \\
V(t^+) &= \left( 1 + \frac{1}{4} \right)V(t), \\
C(t^+) &= C(t), \\
F(t^+) &= F(t) + \frac{1}{8},
\end{align*}
\]

Then \( \omega = 1, \alpha^m = \frac{1}{4}, \alpha^M = \frac{3}{4}, \rho^m = \frac{1}{6}, \rho^M = \frac{1}{2} \). Assume that \([0, 1] \cap \{ t_k \} = \{ t_1, t_2 \} \) and \( \tau \to 0 \), then \( p_1 = p_2 = \frac{1}{4}, q_1 = q_2 = \frac{1}{4} \). By computation, we have \( \sigma_1 = 3 + 72 \ln \frac{5}{4}, \sigma_2 = \frac{9}{2} \sigma_1 - \frac{141}{2} \).

It is easy to show that all conditions of Theorem 1 hold. Thus, (31) has at least one positive 1-periodic solutions. Obviously, \((C_6)\) holds true. If \((C_5)\) holds, then solutions of (31) are globally asymptotically stable.

References


Received: August 5, 2013