On Uni-soft (Quasi) Ideals of AG-groupoids

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Abstract. In this paper we introduced the notions of uni-soft AG-groupoids, uni-soft left(right) ideals and uni-soft quasi ideals of AG-groupoids and also investigated related properties of these structures. Some interesting characterizations of uni-soft AG-groupoids and uni-soft left(right) ideals have been explored.

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1. Introduction

In real world problems, specially in the field of engineering, physics, computer science, medical sciences and social sciences, there are always many uncertainties mixed up in the data. Therefore the classical set theory is not fully suitable for conducting such type of problems. In the past half century, many theories has been developed for dealing such type of problems. For example, in 1965 Zadeh[25] initiated the concept of fuzzy sets, Atanassov[2] gives the notions of intuitionists fuzzy sets, Gorzalzany[7] gives the concept of interval mathematics, and Pawlak[21] initiated the concept of rough sets. The main disadvantage of these theories was the inadequacy of the parameterizations tool.

In 1999, Molodtsov[14] initiated the concept of soft sets as a new mathematical tool for dealing with uncertainties. Later, Maji at al([15],[16],[17])
presented the theory of soft sets and fuzzy soft sets and worked on the applications of soft set theory in decision making problems. The applications of the soft theory in algebraic structures was introduce by Aktas and Cagman[1]. They presented the idea of soft groups as a generalization of the fuzzy groups. Then in 2009, Shabir et al[23] studied soft semigroups and soft ideals over a semigroups. After than, many algebraic properties of soft sets are studied(see [3], [4], [6], [8], [9]).

A groupoid \( S \) is called an Abel-Grassman’s groupoid (AG-groupoid) if the identity: \((ab)c = (cb)a\) holds for all \(a, b, c \in S \) [22]. It was first introduced by Nasseruddin and Kazim in 1972 [11]. They called it left almost semigroup(LA-semigroup). Generally, an AG-groupoid is a non associative structure. It is a midway structure between a groupoid and commutative semigroup. AG-groupoids generalized the concept of commutative semigroup. Although AG-groupoids are not associative but they have a closed link with semigroups. In an AG-groupoid \( S \), the medial law: \((ab)(cd) = (ac)(bd)\) holds \(\forall a, b, c, d \in S\) [5]. In an AG-groupoid \( S \) with left identity, the paramedial law: \((ab)(cd) = (db)(ca)\) holds \(\forall a, b, c, d \in S\) [5]. An AG-groupoid \( S \) is called weak associative or AG*-groupoid if it satisfies the identity \((ab)c = b(ac)\), for all \(a, b, c \in S\) [20]. Also an AG-groupoid is called AG**-groupoid if \(a(bc) = b(ac)\) for all \(a, b, c \in S\).

In 1993, Kamran initiated the notion of AG-group. An AG-groupoid \( S \) is called an AG-group if there exist a left identity \(e \in S\) (i.e \(ea = a, \forall a \in S\)), for all \(a \in S\) there exist \(a^{-1} \in S\) such that \(a^{-1}a = aa^{-1} = e\) [12].

In 2003, Q. Mushtaq [18] discussed ideal in AG-groupoids and in 2010, fuzzification [13] of AG-groupoids were done by M. Khan, M. Noman and A. Khan, which globally attracted the researchers to this field, due to which many papers have published in 2011 and 2012. In 2011, Tariq shah [24] injected the idea of soft set theory in AG-groupoids and introduced the concept of soft ordered AG-groupoid.

Recently, In 2013 C. S. Kim [10] initiated the notions of uni-soft semigroups, uni-soft left(right) ideals and uni-soft quasi ideals of semigroups. The main purpose of this paper is to apply the notions initiated by C. S. Kim [10], to a class of non-associative algebraic structure(i-e AG-groupoid) and derive some related results.

2. Preliminaries

A groupoid \((G, .)\) is called an AG-groupoid if it satisfy the left invertive law: \((ab)c = (cb)a, \forall a, b, c \in G\). A non-empty subset \(H\) of an AG-groupoid \(S\) is called a subAG-groupoid if \(ab \in H\) for all \(a, b \in H\). A non-empty subset \(I\) of an AG-groupoid \(S\) is called a left(right) ideal of \(S\) if \(SI \subseteq I(IS \subseteq I)\). A nonempty subset \(I\) of \(S\) is called an ideal(or a two sided ideal) of \(S\) if it is both left and right ideal of \(S\). A non-empty subset \(Q\) of an AG-groupoid \(S\) is called a quasi ideal of \(S\) if \(SQ \cap QS \subseteq Q\). It is easy to see that every one sided ideal of \(S\) is a quasi ideal
of $S$. An element $a \in S$ is called regular, if there exist some $x \in S$ such that $a = (ax)a, \forall x \in S$. An AG-groupoid $S$ is called regular if every element of $S$ is regular. If $S$ is an AG-groupoid and $A$ and $B$ are any two subset of $S$, then the multiplication of $A$ and $B$ is define by

$$AB = \{ab \in S \mid a \in A \text{ and } b \in B\}.$$ 

**Definition 2.1.** ([4, 14]). Let $U$ be an initial universe and $E$ be a set of parameters. Let $A$ be a non-empty subset of $E$ and $P(U)$ denotes the power set of $U$. A fair $(F, A)$ is called a soft set over $U$, where $F$ is a mapping given by $F : A \to P(U)$. We can say that a soft set is a parameterized family of subset of the universe set $U$. The function $F$ is called approximate function of the soft set $(F, A)$. Clearly, a soft set is not a set.

For any two soft sets $(F, S)$ and $(G, S)$ over a common universe $U$, we say that $(F, S)$ is the soft subset of $(G, S)$ denoted by $(F, S) \subseteq (G, S)$ if $F(x) \subseteq G(x), \forall x \in S$.

The soft union of $(F, S)$ and $(G, S)$ over a common universe $U$ is the soft set $(F \cup G, S)$ over $U$ in which $F \cup G$ is define by

$$(F \cup G)(x) = F(x) \cup G(x), \forall x \in S.$$ 

The soft intersection of $(F, S)$ and $(G, S)$ over $U$ is the soft set $(F \cap G, S)$ over $U$ in which $F \cap G$ is defined by

$$(F \cap G)(x) = F(x) \cap G(x), \forall x \in S.$$ 

Let $(F, A)$ be a soft set over a universe $U$ and $u$ be any subset of $U$, then the $u$-exclusive set of $(F, A)$, denoted by $e_A(F; u)$ is defined to be the set:

$$e_A(F; u) = \{x \in A \mid F(x) \subseteq u\}.$$ 

3. **Uni-soft(Quasi) ideals**

In what follows, we take $E = S$, where $E$ is the set of parameters and $S$ is a non-associative AG-groupoid, unless otherwise specified. We need some definitions which will be useful to derive the main results of the paper.

**Definition 3.1.** Let $(F, S)$ be a soft set over $U$, then $(F, S)$ is called a uni-soft AG-groupoid over $U$ if $F(xy) \subseteq F(x) \cup F(y)$, for all $x, y \in S$.

**Definition 3.2.** Let $(F, S)$ be a soft set over $U$, then $(F, S)$ is called a uni-soft left(right) ideal over $U$ if $F(xy) \subseteq F(x)(F(xy) \subseteq F(x))$, for all $x, y \in S$.

**Definition 3.3.** A soft set $(F, S)$ over $U$ is called a uni-soft two-sided ideal over $U$ if it is both a uni-soft left ideal and a uni-soft right ideal over $U$.

From Definitions 3.1 and 3.2, it is clear that every uni-soft left(right) ideal over $U$ is a uni-soft AG-groupoid over $U$.

**Example 3.1.** Let $S = \{0, 1, 2, 3, 4\}$ be an AG-groupoid with following Cayley table:
Clearly $S$ is a non-associative AG-groupoid, because $(3.3)4 \neq 3.(3.4)$. Now let $(F,S)$ be a soft set over $U$ defined by

$$F : S \to P(U), \ x \mapsto \begin{cases} 
    u_1 & \text{if } x=2, \\
    u_2 & \text{if } x=0, \\
    u_3 & \text{if } x=\{1,4\}, \\
    u_4 & \text{if } x=3.
\end{cases}$$

where $u_1, u_2, u_3, u_4 \in P(U)$ with $u_1 \subset u_2 \subset u_3 \subset u_4$. Then $(F,S)$ is a uni-soft left(right) ideal over $U$ and hence a uni-soft AG-groupoid over $U$.

From the following examples we will see that not every uni-soft AG-groupoid over $U$ is a uni-soft left(right) ideal over $U$.

**Example 3.2.** Let $S = \{0, 1, 2, 3, 4\}$ be an AG-groupoid with following Cayley table:

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</table>

Clearly $S$ is a non-associative AG-groupoid, because $(2.4)0 \neq 2.(4.0)$. Now let $(F,S)$ be a soft set over $U$ defined by

$$F : S \to P(U), \ x \mapsto \begin{cases} 
    u_1 & \text{if } x=0, \\
    u_2 & \text{if } x=1, \\
    u_3 & \text{if } x=4, \\
    u_4 & \text{if } x=2, \\
    u_5 & \text{if } x=3.
\end{cases}$$

where $u_1, u_2, u_3, u_4, u_5 \in P(U)$ with $u_1 \subset u_2 \subset u_3 \subset u_4 \subset u_5$. Then $(F,S)$ is a uni-soft AG-groupoid over $U$ but it is not a uni-soft left ideal over $U$. Since $F(2.0) = F(1) = u_2 \not\subseteq u_1 = F(0)$.

**Example 3.3.** Let $S = \{0, 1, 2, 3, 4\}$ be an AG-groupoid with following Cayley table:
Again $S$ is a non-associative AG-groupoid, because $(1.3).0 \neq 1.(3.0)$. Now let $(F, S)$ be a soft set over $U$ defined by

$$F : S \rightarrow P(U), \quad x \mapsto \begin{cases} u_1 & \text{if } x=2, \\ u_2 & \text{if } x=4, \\ u_3 & \text{if } x=1, \\ u_4 & \text{if } x=\{0,3\}. \end{cases}$$

where $u_1, u_2, u_3, u_4 \in P(U)$ with $u_1 \subset u_2 \subset u_3 \subset u_4$. Then $(F, S)$ is a uni-soft AG-groupoid over $U$ but it is not a uni-soft right ideal over $U$. Since $F(4.1) = F(1) = u_3 \not\subseteq u_2 = F(4)$.

**Theorem 3.1.** If $(F, S)$ and $(G, S)$ are two uni-soft AG-groupoids over $U$, then the soft union $(F \cup G, S)$ is also a uni-soft AG-groupoid over $U$.

**Proof.** Since $(F, S)$ and $(G, S)$ are uni-soft AG-groupoids over $U$, so by definition:

$$F(xy) \subseteq F(x) \cup F(y); \forall x, y \in S, \text{ and } G(xy) \subseteq G(x) \cup G(y); \forall x, y \in S.$$ 

Now,

$$(F \cup G)(xy) = F(xy) \cup G(xy) \subseteq (F(x) \cup F(y)) \cup (G(x) \cup G(y)),$$

$$= (F(x) \cup G(x)) \cup (F(y) \cup G(y)) = (F \cup G)(x) \cup (F \cup G)(y).$$

Hence $(F \cup G, S)$ is a uni-soft AG-groupoid over $U$. 

**Theorem 3.2.** If $(F, S)$ and $(G, S)$ are two uni-soft left(right) ideals over $U$, then the soft union $(F \cup G, S)$ is also a uni-soft left(right) ideal over $U$.

**Proof.** Let $(F, S)$ and $(G, S)$ be any two uni-soft left ideals over $U$, then $F(xy) \subseteq F(y)$ and $G(xy) \subseteq G(y); \forall x, y \in S$. Now,

$$(F \cup G)(xy) = F(xy) \cup G(xy) \subseteq (F(y) \cup G(y)) = (F \cup G)(y)$$

Hence $(F \cup G, S)$ is a uni-soft left ideal over $U$. 

Similarly we can prove that if $(F, S)$ and $(G, S)$ are any two uni-soft right ideals over $U$, then their soft union $(F \cup G, S)$ is also a uni-soft right ideal over $U$.

**Corollary 3.1.** If $(F, S)$ and $(G, S)$ are any two uni-soft two sided ideals over $U$. then their soft union $(F \cup G, S)$ is also a uni-soft two sided ideal over $U$.

**Theorem 3.3.** For any soft set $(F, S)$ over $U$, the following are equivalent:

1. $(F, S)$ is a uni-soft AG-groupoid over $U$.
2. For all $u \in P(U)$ with $e_S(F; u) \neq \phi$, $e_S(F; u)$ is a subAG-groupoid of $S$. 

\[
\begin{array}{c|cccc}
  & 0 & 1 & 2 & 3 & 4 \\
\hline
0 & 0 & 1 & 2 & 3 & 4 \\
1 & 2 & 2 & 2 & 4 & 4 \\
2 & 2 & 2 & 2 & 2 & 2 \\
3 & 3 & 1 & 2 & 0 & 4 \\
4 & 1 & 1 & 2 & 1 & 2 \\
\end{array}
\]
Proof. Let \((F, S)\) is a uni-soft AG-groupoid over \(U\) and \(u \subseteq U\) such that, 
\(e_S(F; u) \neq \phi\). Let \(x, y \in e_S(F; u)\), then \(F(x) \subseteq u\) and \(F(y) \subseteq u\). Since \((F, S)\) is a uni-soft AG-groupoid over \(U\), it follows that 
\[F(xy) \subseteq F(x) \cup F(y) \subseteq u,\]
which implies that \(F(xy) \subseteq u\).

Hence \(xy \in e_S(F; u)\). Thus \(e_S(F; u)\) is a subAG-groupoid of \(S\).

Conversely, Suppose that \(\forall u \in P(U)\) with \(e_S(F; u) \neq \phi\), \(e_S(F; u)\) is a subAG-groupoid of \(S\). \(\eta\) is a subset of \(U\) such that \(F(x) \subseteq u_1\) and \(F(y) \subseteq u_2\). Take \(u = u_1 \cup u_2\), then \(F(x) \subseteq u\) and \(F(y) \subseteq u\), shows that \(x, y \in e_S(F; u)\). But \(e_S(F; u)\) is a subAG-groupoid of \(S\) which implies that \(xy \in e_S(F; u)\). Now 
\[F(xy) \subseteq u = u_1 \cup u_2 = F(x) \cup F(y)\]
Hence \((F, S)\) is a uni-soft AG-groupoid over \(U\).

\[\square\]

In similar manner, we can prove the following Theorem.

**Theorem 3.4.** For any soft set \((F, S)\) over \(U\), the following are equivalent:

1. \((F, S)\) is a uni-soft left(right) ideal over \(U\).
2. For all \(u \in P(U)\) with \(e_S(F; u) \neq \phi\), \(e_S(F; u)\) is a left(right) ideal of \(S\).

The above theorem yield the following corollary.

**Corollary 3.2.** A soft set \((F, S)\) over \(U\) is a uni-soft two sided ideal over \(U\)
if and only if for all \(u \in P(U)\) with \(e_S(F; u) \neq \phi\), \(e_S(F; u)\) is a two sided ideal of \(S\).

Let \((F, S)\) be a soft set over \(U\) and \(u\) be any subset of \(U\) with \(e_S(F; u) \neq \phi\). Define a soft set \((F^*, S)\) over \(U\) as:
\[F^*: S \rightarrow P(U), x \mapsto \begin{cases} F(x) & \text{if } x \in e_S(F; u); \\ \eta & \text{otherwise.} \end{cases}\]
where \(\eta\) is a subset of \(U\) with \(F(x) \subseteq \eta\).

**Theorem 3.5.** If \((F, S)\) is a uni-soft AG-groupoid over \(U\), then \((F^*, S)\) is also a uni-soft AG-groupoid over \(U\).

**Proof.** Let \((F, S)\) is a uni-soft AG-groupoid over \(U\), and \(x, y \in S\). If \(x, y \in e_S(F; u)\), then \(xy \in e_S(F; u)\) as \(e_S(F; u)\) is a subAG-groupoid of \(S\) (by Theorem (3.3)). So
\[F^*(xy) = F^*(x) \cup F^*(y) = F^*(x) \cup F^*(y) .\]
Now, if \(x \) or \(y \) \(\notin e_S(F; u)\), then \(F^*(x) = \eta\) or \(F^*(y) = \eta\). Thus
\[F^*(xy) \subseteq \eta = \eta \cup \eta = F^*(x) \cup F^*(y) .\]
Hence \((F^*, S)\) is a uni-soft AG-groupoid over \(U\).

\[\square\]

**Theorem 3.6.** If \((F, S)\) is a uni-soft left(right) ideal over \(U\), then \((F^*, S)\) is also a uni-soft left(right) ideal over \(U\).
Proof. Let \((F,S)\) is a uni-soft left ideal over \(U\), and \(x,y \in S\) such that \(y \in e_S(F;u)\). Then \(xy \in e_S(F;u)\) as \(e_S(F;u)\) is a left ideal of \(S\) (by Theorem (3.4)). So
\[
F^*(xy) = F(xy) \subseteq F(y) = F^*(y),
\]
which implies that \(F^*(xy) \subseteq F^*(y)\). Now, if \(y \notin e_S(F;u)\), then \(F^*(y) = \eta\). Thus
\[
F^*(xy) \subseteq \eta = F^*(y).
\]
Therefore \((F^*,S)\) is a uni-soft left ideal over \(U\).

Similarly we can show that if \((F,S)\) is a uni-soft right ideal over \(U\), then so is \((F^*,S)\).

Corollary 3.3. If \((F,S)\) is a uni-soft two sided ideal over \(U\), then \((F^*,S)\) is also a uni-soft two sided ideal over \(U\).

For a nonempty subset \(A\) of \(S\), the soft set \((\chi_A,S)\) is called the uni-characteristic soft set where \(\chi_A\) is defined by:
\[
\chi_A : S \rightarrow P(U), \quad x \mapsto \begin{cases} U & \text{if } x \notin A; \\ \phi & \text{if } x \in A. \end{cases}
\]
the soft set \((\chi_S,S)\) is called the uni-empty soft set over \(U\).

Theorem 3.7. For a nonempty subset \(A\) of \(S\), the following are equivalent:

1. \(A\) is a left(right) ideal of \(S\).
2. The uni-characteristic soft set \((\chi_A,S)\) over \(U\) is the uni-soft left(right) ideal over \(U\).

Proof. Suppose \(A\) is a left ideal of \(S\) and \(x,y \in S\). If \(y \in A\), then \(xy \in A\) as \(A\) is a left ideal of \(S\). Therefore \(\chi_A(xy) = \phi = \chi_A(y)\).

Now, if \(y \notin A\), then \(\chi_A(xy) \subseteq U = \chi_A(y)\). Hence \((\chi_A,S)\) is a uni-soft left ideal over \(U\). Similarly we can show that if \(A\) is a right ideal of \(S\), then \((\chi_A,S)\) is a uni-soft right ideal over \(U\).

Conversely, assume that \((\chi_A,S)\) is uni-soft left ideal over \(U\), we will show that \(A\) is a left ideal of \(S\). For this let \(x \in S\) and \(y \in A\). Since \((\chi_A,S)\) is uni-soft left ideal over \(U\), so by definition we have \(\chi_A(xy) \subseteq \chi_A(y)\). But \(\chi_A(y) = \phi\) as \(y \in A\). Therefore
\[
\chi_A(xy) \subseteq \chi_A(y) = \phi.
\]
which implies that \(xy \in A\) and hence \(A\) is a left ideal of \(S\). Similarly we can prove that if \((\chi_A,S)\) is a uni-soft right ideal over \(U\), then \(A\) is a right ideal of \(S\).

Corollary 3.4. For a nonempty subset \(A\) of \(S\), the following assertions are equivalent:

1. \(A\) is a two sided ideal of \(S\).
2. The uni-characteristic soft set \((\chi_A,S)\) over \(U\) is the uni-soft two sided ideal over \(U\).
Let \((F, S)\) and \((G, S)\) be any two soft sets over \(U\), then the uni-soft product \((F, S)\) and \((G, S)\) is defined by \((F \circ G, S)\) over \(U\) where \(F \circ G\) is a mapping from \(S\) to \(P(U)\) given by:

\[
(F \circ G)(x) = \begin{cases} 
\bigcap_{x = yz} \{F(y) \cup G(z)\} & \text{if } \exists y, z \in S \text{ such that } x = yz; \\
U & \text{otherwise.}
\end{cases}
\]

**Theorem 3.8.** A soft set \((F, S)\) over \(U\) is a uni-soft AG-groupoid over \(U\) if and only if \((F, S) \subseteq (F \circ F, S)\).

**Proof.** Suppose that \((F, S)\) is a uni-soft AG-groupoid over \(U\), then \(F(x) \subseteq F(y) \cup F(z); \forall x \in S\), with \(x = yz\). Now,

\[
F(x) \subseteq \bigcap_{x = yz} \{F(y) \cup F(z)\} = (F \circ F)(x); \forall x \in S.
\]

Hence \((F, S) \subseteq (F \circ F, S)\).

Conversely, let \((F, S) \subseteq (F \circ F, S)\), and let \(x, y \in S\), then \(xy \in S\) as \(S\) is a groupoid. Now,

\[
F(xy) \subseteq (F \circ F)(xy) \subseteq F(x) \cup F(y),
\]

which shows that \((F, S)\) is a uni-soft AG-groupoid over \(U\). \(\square\)

**Proposition 3.1.** For the soft sets \((F_1, S), (F_2, S), (G_1, S)\) and \((G_2, S)\) over \(U\), if \((F_1, S) \subseteq (G_1, S)\) and \((F_2, S) \subseteq (G_2, S)\), then

\[
(F_1 \circ F_2, S) \subseteq (G_1 \circ G_2, S).
\]

**Proof.** Suppose that \(x \in S\) and \(x\) is not be expressed as \(x = yz\) for some \(y, z \in S\). Then

\[
(F_1 \circ F_2)(x) = U = (G_1 \circ G_2)(x),
\]

which implies \((F_1 \circ F_2, S) \subseteq (G_1 \circ G_2, S)\).

Now, if there exist some \(y, z \in S\) such that \(x = yz\), then

\[
(F_1 \circ F_2)(x) = \bigcap_{x = yz} \{F_1(y) \circ F_2(z)\} \subseteq \bigcap_{x = yz} \{G_1(y) \circ G_2(z)\} = (G_1 \circ G_2)(x).
\]

Hence \((F_1 \circ F_2, S) \subseteq (G_1 \circ G_2, S)\). \(\square\)

**Theorem 3.9.** For any soft set \((G, S)\) and uni-empty soft set \((\chi_S, S)\) over \(U\), the following are equivalent:

1. \((G, S) \subseteq (\chi_S \circ G, S)\).
2. \((G, S)\) is a uni-soft left ideal over \(U\).

**Proof.** Suppose \((G, S) \subseteq (\chi_S \circ G, S)\) and \(x, y \in S\). Then

\[
G(xy) \subseteq (\chi_S \circ G)(xy) \subseteq \chi_S(x) \cup G(y) = \phi \cup G(y) = G(y),
\]

shows that \((G, S)\) is a uni-soft left ideal over \(U\).

Conversely, let \((G, S)\) is a uni-soft left ideal over \(U\) and let \(x, y \in S\). If \(x = yz\) for some \(y, z \in S\). Then
\[(\chi_S \circ G)(x) = \bigcap_{x=yz} \{\chi_S(y) \cup G(z)\} \supseteq \bigcap_{x=yz} \{\phi \cup G(yz)\} = \bigcap_{x=yz} \{G(yz)\} = G(x)\]

Otherwise, \(G(x) \subseteq U = (\chi_S \circ G)(x)\), for all \(x \in S\). Hence \((G, S) \subseteq (\chi_S \circ G, S)\).

**Theorem 3.10.** For any soft set \((G, S)\) and uni-empty soft set \((\chi_S, S)\) over \(U\), the following are equivalent:

1. \((G, S) \subseteq (G \circ \chi_S, S)\).
2. \((G, S)\) is a uni-soft right ideal over \(U\).

**Proof.** It is the same as the proof of Theorem(3.9). \(\square\)

**Corollary 3.5.** For any soft set \((G, S)\) and uni-empty soft set \((\chi_S, S)\) over \(U\), the following are equivalent:

1. \((G, S) \subseteq (\chi_S \circ G, S)\) and \((G, S) \subseteq (G \circ \chi_S, S)\).
2. \((G, S)\) is a uni-soft two sided ideal over \(U\).

**Theorem 3.11.** Let \((F, S)\) and \((G, S)\) be two soft sets over \(U\) where \(S\) is an AG\(^*\)-groupoid. If \((F, S)\) is a uni-soft right ideal over \(U\), then \((F \circ G, S)\) is a uni-soft left ideal over \(U\).

**Proof.** Let \(x, y \in S\) and \(y\) can be expressed as \(y = ab\) for some \(a, b \in S\). Then \(xy = x(ab) = (ax)b\), for all \(a, b, x \in S\), because \(S\) is an AG\(^*\)-groupoid. Now

\[
(F \circ G)(y) = \bigcap_{y=ab} \{F(a) \cup G(b)\} \supseteq \bigcap_{xy=(ax)b} \{F(ax) \cup G(b)\} = \bigcap_{xy=cb} \{F(c) \cup G(b)\} = (F \circ G)(xy),
\]

which implies that \((F \circ G)(xy) \subseteq (F \circ G)(y)\). Hence \((F \circ G, S)\) is a uni-soft left ideal over \(U\). \(\square\)

**Theorem 3.12.** Let \((F, S)\) is a uni-soft right ideal over \(U\) and \((G, S)\) is a uni-soft left ideal over \(U\). Then

\[
(F \circ G, S) \supseteq (F \cup G, S).
\]

**Proof.** Let \((F, S)\) and \((G, S)\) be any two uni-soft right ideal and uni-soft left ideal over \(U\), respectively. Also let \(x \in S\) such that \(x = ab\) for some \(a, b \in S\). Then

\[
(F \circ G)(x) = \bigcap_{x=ab} \{F(a) \cup G(b)\} \supseteq \bigcap_{x=ab} \{F(ab) \cup G(ab)\} = F(x) \cup G(x) = (F \cup G)(x).
\]

Now, if \(x\) cannot be expressed as \(x = ab\) for some \(a, b \in S\). Then

\[
(F \circ G)(x) = U \supseteq (F \cup G)(x).
\]

Hence \((F \circ G, S) \supseteq (F \cup G, S)\). \(\square\)
Theorem 3.13. For a regular AG-groupoid $S$, if $(F, S)$ is a uni-soft right ideal over $U$ and $(G, S)$ be any soft set over $U$. Then

$$(F \circ G, S) \subseteq (F \cup G, S).$$

Proof. Let $(F, S)$ be a uni-soft right ideal over $U$ and $x \in S$. Since $S$ is regular, so there exist some $a \in S$ such that $x = (xa)x$. Now,

$$(F \cup G)(x) = F(x) \cup G(x) \supseteq F(xa) \cup G(x),$$

as $(F, S)$ is a uni-soft right ideal over $U$.

Since $x = (xa)x$, So

$$(F \circ G)(x) = \bigcap_{y \in S} \{F(y) \cup G(z)\} \subseteq F(xa) \cup G(x) \subseteq (F \cup G)(x).$$

Thus $(F \circ G)(x) \subseteq (F \cup G)(x), \forall x \in S$. Hence $(F \circ G, S) \subseteq (F \cup G, S)$. □

Theorem 3.14. Let $S$ be a regular AG-groupoid. Then for any uni-soft right ideal $(F, S)$ and uni-soft left ideal $(G, S)$ over $U$, we have

$$(F \circ G, S) = (F \cup G, S).$$

Proof. Suppose that $S$ is a regular AG-groupoid and $(F, S)$ and $(G, S)$ are any two uni-soft right and uni-soft left ideals over $U$, respectively. Then by Theorem (3.12), we have:

$$(F \circ G, S) \supseteq (F \cup G, S).$$

Also by Theorem (3.13), we have:

$$(F \circ G, S) \subseteq (F \cup G, S).$$

Hence we will get $(F \circ G, S) = (F \cup G, S)$. □

Definition 3.4. A soft set $(H, S)$ over $U$ is called a uni-soft quasi ideal over $U$ if

$$(H \circ \chi_S, S) \cup (\chi_S \circ H, S) \supseteq (H, S).$$

In general, every uni-soft left(right) ideal is a uni-soft quasi ideal over $U$, but the converse does not hold as we see in the following example.

Example 3.4. Let $S = \{1, 2, 3, 4, 5\}$ be an AG-groupoid with following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>

We see that $S$ is a non-associative AG-groupoid, because $(3,3).4 \neq 3.(3,4)$.

Now let $(F, S)$ be a soft set over $U$ defined by
F : S → P(U), x ↦ { \phi if x=5,
                     u if x=\{1,2,3,4\},

where u ∈ P(U) such that u ≠ \phi. Then (F,S) is a uni-soft quasi ideal over U, but it is not a uni-soft left(right) ideal over U. Because F(1.5) = F(4) = u \not\subset \phi = F(5), shows that (F,S) is not a uni-soft left ideal over U. Also F(5.1) = F(4) = u \not\subset \phi = F(5), implies that (F,S) is not a uni-soft right ideal over U.

References


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