Basic Theorem for Generalized One-sided Concept Lattices

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Abstract

The Basic Theorem for Concept Lattices represents one of the fundamental tool for a theoretical study of concept lattices. In this paper the similar assertion for generalized one-sided concept lattices is derived.

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1 Introduction

The generalized one-sided concept lattices [4] represent a generalization of one-sided concept lattices, cf. [3, 8], convenient for analysis of object-attribute models with different truth value structures. The main aim of this paper is to present characterization of complete lattices as generalized one-sided concept lattices. As in the other cases, this characterization is given by so-called Basic Theorem on Concept Lattices, cf. [7], [1, 2] or [9]. It provides necessary and sufficient condition on formal context under which the corresponding concept lattice is isomorphic to a given complete lattice. In order to obtain Basic Theorem for generalized one-sided concept lattices we describe a representation of generalized one-sided concept lattices in the framework of FCA. Using this representation we provide a relatively simple proof of our Basic Theorem.

First we briefly recall the basic notions of FCA. Let $(G, M, I)$ be a formal context, i.e., $G, M \neq \emptyset$ and $I \subseteq G \times M$. There is a pair of mappings $\uparrow: \mathcal{P}(G) \rightarrow \mathcal{P}(M)$ and $\downarrow: \mathcal{P}(M) \rightarrow \mathcal{P}(G)$, which forms a Galois connection between power sets of $G$ and $M$ respectively.

$$X^\uparrow = \{y \in G : (x, y) \in I, \forall x \in X\},$$

$$Y^\downarrow = \{x \in M : (x, y) \in I, \forall y \in Y\}.$$

The corresponding concept lattice is denoted by $\mathfrak{B}(G, M, I)$.

There is so-called the Basic Theorem on Concept Lattices, which gives the equivalent conditions for a complete lattice to be isomorphic with a given concept lattice, cf. [7].

**Theorem 1.1.** A complete lattice $V$ is isomorphic to $\mathfrak{B}(G, M, I)$ if and only if there are mappings $\bar{\gamma}: G \rightarrow V$ and $\bar{\mu}: M \rightarrow V$ such that $\bar{\gamma}(G)$ is supremum-dense in $V$, $\bar{\mu}(M)$ is infimum-dense in $V$ and $(g, m) \in I$ is equivalent to $\bar{\gamma}(g) \leq \bar{\mu}(m)$ for all $g \in G$ and all $m \in M$.

Next we give a definition of generalized one-sided concept lattice which generalizes the notion of one-sided concept lattice, cf [4]. First we recall the definition of a generalized one-sided formal context.

A 4-tuple $(B, A, \mathcal{L}, R)$ is said to be a generalized one-sided formal context if the following conditions are fulfilled:

i) $B$ is a non-empty set of objects and $A$ is a non-empty set of attributes.

ii) $\mathcal{L}: A \rightarrow \mathcal{CL}$, $\mathcal{CL}$ denotes the class of all complete lattices.

iii) $R: B \times A \rightarrow \bigcup_{a \in A} \mathcal{L}(a)$ is a mapping satisfying $R(b, a) \in \mathcal{L}(a)$ for all $b \in B$ and $a \in A$. 
Let us note that for any attribute \( a \), \( \mathcal{L}(a) \) denotes the complete lattice, which represents a lattice structure of truth values for the attribute \( a \). The symbol \( R \) denotes so-called (generalized) incidence relation, where \( R(b, a) \) represents a degree from the structure \( \mathcal{L}(a) \) in which the element \( b \in B \) has the given attribute \( a \).

As in the case of concept lattices, there are concept forming operators which form a Galois connection between \( \mathcal{P}(B) \) and the direct product \( \prod_{a \in A} \mathcal{L}(a) \).

Let \((B, A, \mathcal{L}, R)\) be a generalized one-sided formal context. We define a pair of mappings \( \uparrow : \mathcal{P}(B) \to \prod_{a \in A} \mathcal{L}(a) \) and \( \downarrow : \prod_{a \in A} \mathcal{L}(a) \to \mathcal{P}(B) \) as follows:

\[
\uparrow(X)(a) = \bigwedge_{b \in X} R(b, a), \quad \text{for all } X \subseteq B, \quad (1)
\]

\[
\downarrow(g) = \{ b \in B : \forall a \in A, \ g(a) \leq R(b, a) \}, \quad \text{for all } g \in \prod_{a \in A} \mathcal{L}(a). \quad (2)
\]

With the help of these two operators we can define a set of fixed points

\[ \mathfrak{S}(B, A, \mathcal{L}, R) = \{(X, g) : \uparrow(X) = g, \downarrow(g) = X \} \]

Further we define partial order on the set \( \mathfrak{S}(B, A, \mathcal{L}, R) \) as follows:

\[
(X_1, g_1) \leq (X_2, g_2) \iff X_1 \subseteq X_2 \iff g_1 \geq g_2. \quad (3)
\]

The set \( \mathfrak{S}(B, A, \mathcal{L}, R) \) with this partial ordering forms a complete lattice, which is called generalized one-sided concept lattice (see 4 for more details).

As it can be shown, the generalized one-sided concept lattices represent special case of fuzzy concept lattices, cf [10, 11, 12].

## 2 Main Result

In order to prove the Basic Theorem for generalized one-sided concept lattices, we provide a representation of generalized one-sided concept lattices in the framework of classical concept lattices. Let \((B, A, \mathcal{L}, R)\) be a generalized one-sided formal context and \( J_a \) be a supremum-dense subset of the lattice \( \mathcal{L}(a) \), for all \( a \in A \). We put \( S_a = \{ w_a : w \in J_a \} \) and we define a new set of attributes \( S = \bigcup_{a \in A} S_a \).

Finally, we define an incidence relation \( I \subseteq B \times S \) by

\[
(b, w_a) \in I \quad \text{iff} \quad w \leq R(b, a). \quad (4)
\]

Since both pairs of operators \( \uparrow, \downarrow \) and \( \uparrow, \downarrow \), form Galois connections, the sets \( C_I = \{ Y^\uparrow : Y \subseteq S \} \) and \( C_R = \{ \downarrow(g) : g \in \prod_{a \in A} \mathcal{L}(a) \} \) form closure systems on the set of all objects \( B \). Using similar methods as in [5] and [6] we prove the following result.
Theorem 2.1. Let \((B, S, I)\) be the formal context associated to a generalized one-sided formal context \((B, A, \mathcal{L}, R)\). Then the closure systems \(C_I\) and \(C_R\) on the set of all objects \(B\) are identical.

Proof. First we show that \(C_I \subseteq C_R\). Let \(Y \subseteq S\) be an arbitrary subset. We show that \(Y^\downarrow = \downarrow (g)\) for some \(g \in \prod_{a \in A} \mathcal{L}(a)\). We put \(Y_a = \{w \in J_a : w \in Y\}\).

Next we define \(g : A \to \bigcup_{a \in A} \mathcal{L}(a)\) by \(g(a) = \bigvee Y_a = \bigvee \{w \in J_a : w \in Y\}\).

Obviously \(g(a) \in \mathcal{L}(a)\), thus \(g \in \prod_{a \in A} \mathcal{L}(a)\). According to basic properties of the supremum and the condition (4), for all \(b \in B\) and \(a \in A\) we obtain

\[
g(a) = \bigvee Y_a \leq R(b, a) \iff \{w \in B : \forall a \in A, (b, w_a) \in I, \forall w \in Y_a\}
\]

Since for any fixed element \(b \in B\) holds: \((b, w_a) \in I\) for all \(w_a \in Y\) if and only if for all \(a \in A\), \((b, w_a) \in I\), for all \(w \in Y_a\) we obtain

\[
\downarrow (g) = \{b \in B : \forall a \in A, g(a) \leq R(b, a)\} = \{b \in B : \forall a \in A, (b, w_a) \in I, \forall w \in Y_a\} = \{b \in B : (b, w_a) \in I, \forall w \in Y\} = Y^\downarrow.
\]

Hence for any \(Y \subseteq S\) we have \(Y^\downarrow \subseteq C_R\), which yields \(C_I \subseteq C_R\).

Conversely, we show the opposite inclusion. Let \(g \in \prod_{a \in A} \mathcal{L}(a)\) be an arbitrary element. Since the set \(J_a\) is supremum-dense in \(\mathcal{L}(a)\) for each \(a \in A\) we can express an element \(g(a)\) as \(g(a) = \bigvee Y_a\) for some subset \(Y_a \subseteq J_a\). We put \(Y = \bigcup_{a \in A} \{w_a : w \in Y_a\}\).

As in the previous case we obtain for all \(b \in B\) and all \(a \in A\)

\[
g(a) \leq R(b, a) \iff (b, w_a) \in I, \forall w \in Y_a,
\]

which yields \(\downarrow (g) = Y^\downarrow\). This gives \(C_R \subseteq C_I\), what completes the proof. \(\square\)

Since any closure system on the set of objects uniquely determines the order structure of the corresponding concept lattices, we obtain that \(\mathcal{G}(B, A, \mathcal{L}, R)\) and \(\mathcal{G}(B, S, I)\) are isomorphic.

Now we can provide the characterization of generalized one-sided concept lattices.

Theorem 2.2. Let \((B, A, \mathcal{L}, R)\) be a generalized one-sided formal context and for all \(a \in A\) \(J_a\) denotes a supremum-dense subset of \(\mathcal{L}(a)\). A complete lattice \(V\) is isomorphic to \(\mathcal{G}(B, A, \mathcal{L}, R)\) if and only if there is a mapping \(\gamma : B \to V\) and for all \(a \in A\) there are mappings \(\mu_a : J_a \to V\) such that \(\gamma(B)\) is supremum-dense in \(V\), \(\bigcup_{a \in A} \mu_a(J_a)\) is infimum-dense in \(V\) and for all \(b \in B\), \(a \in A\) and \(w \in J_a\) condition \(w \leq R(b, a)\) is equivalent to \(\gamma(b) \leq \mu(w)\).
Proof. Let \((B, A, \mathcal{L}, R)\) be a generalized one-sided formal context, for each \(a \in A\), \(J_a\) be a supremum-dense subset of the lattice \(\mathcal{L}(a)\) and \(V\) be a complete lattice such that \(V \cong \mathfrak{C}(B, A, \mathcal{L}, R)\). According to the Theorem 2.1, the lattice \(\mathfrak{C}(B, A, \mathcal{L}, R)\) is isomorphic to the concept lattice \(\mathfrak{B}(B, S, I)\) where \(S = \bigcup_{a \in A} \{w_a : w \in J_a\}\) and \(w \leq R(b, a)\) if and only if \((b, w_a)\) \(\in I\). Consequently, \(\mathfrak{B}(B, S, I) \cong V\) and we are able to use Theorem 1.1. There are mappings \(\gamma : B \to V\) and \(\mu : S \to V\) such that \(\gamma(B)\) is supremum-dense in \(V\), \(\mu(G)\) is infimum-dense in \(V\) and \((g, m) \in I\) is equivalent to \(\gamma(g) \leq \mu(m)\) for all \(g \in G\) and all \(m \in M\). Hence we define \(\gamma : B \to V\) as \(\gamma(b) = \gamma(b)\) for all \(b \in B\) and we define for all \(a \in A\) mappings \(\mu_a : J_a \to V\) as \(\mu_a(w_a) = \mu(w_a)\). Evidently, \(\gamma(B)\) is supremum-dense in \(V\), as well as \(\bigcup_{a \in A} \mu(J_a) = \mu(S)\) is infimum-dense in \(V\). From the condition (4) we obtain for all \(b \in B\), for all \(a \in A\) and for all \(w \in J_a\)

\[
w \leq R(b, a) \iff (b, w_a) \in I \iff \gamma(b) \leq \mu(w_a) \iff \gamma(b) \leq \mu_a(w).
\]

Conversely, assume that there are mappings \(\gamma : B \to V\) and \(\mu_a : J_a \to V\) for all \(a \in A\), satisfying condition of Theorem 2.2. Under the same representation of \(\mathfrak{C}(B, A, \mathcal{L}, R)\) as \(\mathfrak{B}(B, S, I)\), we show that \(V \cong \mathfrak{B}(B, S, I)\).

We define \(\gamma(b) = \gamma(b)\) for all \(b \in B\) and we define \(\mu(w_a) = \mu_a(w)\) for all \(a \in A\) and for all \(w \in J_a\). Again \(\gamma(B)\) is supremum-dense in \(V\) and \(\mu(S) = \bigcup_{a \in A} \mu(J_a)\) is infimum dense in \(V\). Moreover for all \(b \in B\), \(a \in A\) and for all \(w \in J_a\)

\[
(b, w_a) \in I \iff w \leq R(b, a) \iff \gamma(b) \leq \mu_a(w) \iff \gamma(b) \leq \mu(w_a),
\]

hence, due to Theorem 2.1 and Theorem 1.1, we obtain \(\mathfrak{C}(B, A, \mathcal{L}, R) \cong \mathfrak{B}(B, S, I) \cong V\).

\[\square\]

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