Legendre’s Equation Expressed by the Initial Value by Using Integral Transforms

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Abstract

In this article, we have proposed the solution of Legendre’s equation expressed by the initial value by using the Laplace transform, and the proposed method should be applied to another equations. Additionally, we have checked representations of the equation by using the Sumudu transform and Elzaki’s.

Mathematics Subject Classification: 44A10, 34B30

Keywords: Legendre’s equation, integral transform, variable coefficients

1 Introduction

Legendre’s differential equation and Legendre function are likely to occur in problems showing spherical symmetry, and they carry out a role in the expansion of the Newtonian potential and boundary value problems for spheres such
as Laplace’s equation in spherical coordinates[20]. The equation has the form of

$$\left(1 - t^2\right)y'' - 2ty' + n(n + 1)y = 0,$$

where \(n\) is a given constant, and this is so-called the big equation of physics. Normally, using tool to get the solution of it is the power series method. But in this article, we would like to propose the solution of the equation expressed by the initial value by using the Laplace transform. Subordinately, we have expressed the solution of Legendre’s equation based on Laplace transform using convolution, and similarly checked representations of the equation by using the Sumudu transform and Elzaki’s.

The proposed research has been pursued with intention to try the solution of differential equations with variable coefficients by using integral transforms, and surely, this method can be applied to another equation such as Bessel’s, Euler-Cauchy, and Sturm-Liouville’s. In this article, mainly used tool is the Laplace transform, and using tool should be exchangeable to another integral transforms such as the Sumudu one or Elzaki’s.

We would like to take a look into preceding researches of this area. Watugula introduced the Sumudu transform in order to apply it to find the solution of ordinary differential equations in controlling engineering problems in 1993[24]. Agwa dealt with the Sumudu transform on time scales in [1], and there are many researches related to the transform[1-3, 13]. Recently(2011) Elzaki proposed the Elzaki transform[7-11] defined by

$$T(u) = u \int_{0}^{\infty} e^{-t/u} f(t) dt$$

for \(E[f(t)] = T(u)\), and he says that it efficiently be used to solve problems without resorting to anew frequency domain because it preserve scales and units properties[8, 15]. We have checked it to find the solution of differential equations with variable coefficients[16, 19], and have found it has a strong point managing differential equations with variable coefficients when compared with existing other integral transforms. On the other hand, efforts to find solutions of differential equations with variable coefficients by using integral transforms have been pursued[5-6, 14-16, 18-25], and there are many researches related to differential equations with variable coefficients[4, 12, 26].

We have showed that the solution of Legendre’s equation by using Laplace transform can be expressed by

$$y = \mathcal{L}^{-1}(Y) = \left[\frac{n+1}{2n+1} \cdot \cos \sqrt{n} t + \frac{n}{2n+1} \cdot \cos \sqrt{n+1} t\right] \cdot y(0)$$

$$+ \left[\frac{n-1}{2n+1} \cdot \frac{1}{\sqrt{n}} \cdot \sinh \sqrt{n} t + \frac{n+2}{2n+1} \cdot \frac{1}{\sqrt{n+1}} \cdot \sin \sqrt{n+1} t\right] \cdot y'(0)$$
where \( h \) is a hyperbolic function and \( n \) is a given constant, or alternately,
\[
-ty = \{(1 + \frac{1}{2}n(n + 1)t^2) * y\} + y(0)t - \frac{1}{2}y'(0)t^2,
\]
where * is the standard notation of convolution. The calculation with respect to this is not difficult and simple.

## 2 Legendre’s equation expressed by the initial value by using integral transforms

The Laplace transform method is performing a worthy role for solving linear ordinary differential equations and corresponding initial value problems, having an idea to replace operations of calculus on functions with operations of algebra on transforms. In this section, we would like to propose the solution of Legendre’s equation expressed by the initial value based on Laplace transform. From the table of Laplace transforms\[20\], note that
\[
\mathcal{L}(\frac{1}{w}\sin wt) = \frac{1}{s^2 + w^2},
\]
\[
\mathcal{L}(\cos wt) = \frac{s}{s^2 + w^2},
\]
\[
\mathcal{L}(\cos h at) = \frac{s}{s^2 - a^2}
\]
and
\[
\mathcal{L}(\frac{1}{a}\sin h at) = \frac{1}{s^2 - a^2}
\]
where \( h \) is a hyperbolic function.

**Theorem 2.1** The solution of Legendre’s equation
\[
(1 - t^2)y'' - 2ty' + n(n + 1)y = 0
\]
can be expressed by
\[
y = \mathcal{L}^{-1}(Y) = \left[ \frac{n + 1}{2n + 1} \cdot \cos h\sqrt{n} t + \frac{n}{2n + 1} \cdot \cos \sqrt{n + 1} t \right] \cdot y(0)
\]
\[
+ \left[ \frac{n - 1}{2n + 1} \cdot \frac{1}{\sqrt{n}} \cdot \sinh \sqrt{n} t + \frac{n + 2}{2n + 1} \cdot \frac{1}{\sqrt{n + 1}} \cdot \sin \sqrt{n + 1} t \right] \cdot y'(0)
\]
by using the Laplace transform, where \( h \) is a hyperbolic function and \( n \) is a given constant.
Proof. To begin with, let us expand the equation, and we have
\[ y'' - t^2 y'' - 2ty' + n(n + 1)y = 0. \]

Taking the Laplace transform on both sides of Legendre’s equation, we have
\[
(s^2 Y - sy(0) - y'(0)) - (s^2 d^2 Y/ds^2 + 4s dY/ds + 2Y) \\
-2(-Y - s dY/ds) + n(n + 1)Y = 0
\]
for \( Y = \mathcal{L}(y) = F(s) \). Arranging the equation, we have
\[
s^2 d^2 Y/ds^2 + 2s dY/ds - \{ s^2 + n(n + 1) \} Y + sy(0) + y'(0) = 0.
\]

Since \( Y'' = s^2 Y - sy(0) - y'(0) \) and \( Y' = sY - y(0) \), we have
\[
s^2 \{ s^2 Y - sy(0) - y'(0) \} + 2s(sY - y(0)) \\
-\{ s^2 + n(n + 1) \} Y + sy(0) + y'(0) = 0.
\]

Organizing the equality, we get
\[
Y = \frac{s^3 + s}{s^4 + s^2 - n(n + 1)} y(0) + \frac{s^2 - 1}{s^4 + s^2 - n(n + 1)} y'(0)
\]
for \( s^4 + s^2 - n(n + 1) \neq 0 \), and note that \( s^4 + s^2 - n(n + 1) = (s^2 - n)(s^2 + (n + 1)). \)

Using partial fractions, we obtain
\[
Y = \left[ \frac{n+1}{2n+1} s - \frac{1}{2n+1} \right] \cdot y(0) + \left[ \frac{n}{2n+1} s - \frac{1}{2n+1} \right] \cdot y'(0).
\]

As we scan a table of Laplace transforms, we see that
\[
y = \mathcal{L}^{-1}(Y) = \left[ \frac{n+1}{2n+1} \cdot \cos h \sqrt{n} \, t + \frac{n}{2n+1} \cdot \cos \sqrt{n+1} \, t \right] \cdot y(0)
\]
\[
+\left[ \frac{n-1}{2n+1} \cdot \frac{1}{\sqrt{n}} \cdot \sinh \sqrt{n} \, t + \frac{n+2}{2n+1} \cdot \frac{1}{\sqrt{n+1}} \cdot \sin \sqrt{n+1} \, t \right] \cdot y'(0)
\]
for \( h \) is a hyperbolic function.

**Corollary 2.2** Legendre’s equation
\[
(1 - t^2)y'' - 2ty' + n(n + 1)y = 0
\]
can be expressed by the convolution of \( y \) and \( 1 + \frac{1}{2}n(n + 1)t^2 \) as
\[
-ty = \{ (1 + \frac{1}{2}n(n + 1)t^2) * y \} + y(0)t - \frac{1}{2}y'(0)t^2
\]
where * is the standard notation of convolution.
Legendre’s equation expressed by the initial value

Proof. Note that $\mathcal{L}(ty) = -F'(s)$ for $Y = \mathcal{L}(y) = F(s)$. Substituting $\mathcal{L}(y') = sy - y(0)$ and $\mathcal{L}(y'') = s^2Y - sy(0) - y'(0)$ into Legendre’s equation, it can be expressed by

$$
(s^2Y - sy(0) - y'(0)) - \frac{d^2}{ds^2}(s^2Y - sy(0) - y'(0))
+ 2\frac{d}{ds}(sY - y(0)) + n(n + 1)Y = 0.
$$

Differentiating the second term and collecting the $Y$-terms, we have

$$
s^3\frac{dY}{ds} = \{s^2 + n(n + 1)\}Y + sy(0) - y'(0).
$$

Hence

$$
\frac{dY}{ds} = \left\{\frac{1}{s} + \frac{n(n + 1)}{s^3}\right\}Y + \frac{y(0)}{s^2} - \frac{y'(0)}{s^3}.
$$

Taking the inverse Laplace transform on both sides, we have

$$
-ty = \{(1 + \frac{1}{2}n(n + 1)t^2) \ast y\} + y(0)t - \frac{1}{2}y'(0)t^2.
$$

Since Legendre’s, Bessel’s and other ordinary differential equations can be written as a Sturm-Liouville’s, we can this idea apply to Bessel’s and Sturm-Liouville’s. For example, let us consider the equation

$$
y'' + \nu^2y = 0, \ y(0) = 0, \ y'(0) = 1,
$$

where $\nu$ is a given number. We can easily get the general solution

$$
y(t) = \frac{1}{\nu} \cdot \sin \nu t, \quad (a)
$$

where $\nu = 1, 2, \cdots$. Next, let us take the Laplace transform on both sides, and apply to the method of theorem 2.1 on given equation. Then we have

$$
s^2Y - sy(0) - y'(0) + \nu^2Y = 0
$$

for $Y = \mathcal{L}(y) = F(s)$. Organizing the equation, we have the answer

$$
y = \frac{\sin \nu t}{\nu}
$$

for $\nu$ is a given number. Surely, this is the same result with (a)[16]. The Sumudu transform and Elzaki’s are simple variants of the Laplace transform and its difference is only that of kernel. In a similar fashion, we would like to
Yechan Song and Hwajoon Kim propose representations of Legendre’s equation by using the Sumudu transform and Elzaki’s. First, we would like to check the case of Elzaki representation. Let \( E[f(t)] = T(u) \) and let \( T^{(n)}(u) \) is the Elzaki transform of the derivative of \( f(t) \). Elzaki showed that

\[
T^{(n)}(u) = \frac{T(u)}{u^n} - \sum_{k=0}^{n-1} u^{2-n+k} f^{(k)}(0)
\]

for \( n \geq 1 \) and

\[
E[tf(t)] = u^2 \frac{dT(u)}{du} - uT(u)
\]

for \( E[f(t)] = T(u) \). Clearly, we can naturally obtain the following results from the definition of it and simple calculations.

1) \( E[f'(t)] = T(u)/u - uf(0) \)

2) \( E[f''(t)] = T(u)/u^2 - f(0) - uf'(0) \)

3) \( E[tf'(t)] = u^2 \frac{d}{du}[T(u)/u - uf(0)] - u[T(u)/u - uf(0)] \)

4) \( E[t^2 f'(t)] = u^4 \frac{d^2}{du^2}[T(u)/u - uf(0)] \)

5) \( E[t^2 f''(t)] = u^2 \frac{d}{du}[T(u)/u^2 - f(0) - uf'(0)] - u[T(u)/u^2 - f(0) - uf'(0)] \)

6) \( E[t^2 f''(t)] = u^4 \frac{d^2}{du^2}[T(u)/u^2 - f(0) - uf'(0)]. \)

7) \( E[t^n] = n! \ u^{n+2} \)

for \( E(f(t)) = T(u)[8] \).

**Theorem 2.3** Legendre’s equation

\[
(1 - t^2)y'' - 2ty' + n(n + 1)y = 0
\]

should be expressed by

\[
T(u) = \frac{u^4 + u^2}{(n^2 + n - 1)u^2 + 1} \ y(0) + \frac{u^3 - u^5}{(n^2 + n - 1)u^2 + 1} \ y'(0)
\]

by using the Elzaki transform for \( n \) is a given constant and \( E(f(t)) = T(u) \).

Proof. Expanding the equation, we have

\[
y'' - t^2 y'' - 2ty' + n(n + 1)y = 0.
\]
Taking the Elzaki transform on both sides of Legendre’s equation, we have

\[ T(u)/u^2 - y(0) - uy'(0) - u^4 \frac{d^2}{du^2}[T(u)/u^2 - y(0) - uy'(0)] \]

\[-2[u^2 \frac{d}{du}\{T(u)/u - uy(0)\} - u\{T(u)/u - uy(0)\}] + n(n+1)T(u) = 0 \]

for \( E(f(t)) = T(u) \). Arranging the equation, we have

\[ u^2T''(u) - 2uT'(u) - \{n(n+1) - 2 + \frac{1}{u^2}\}T(u) + y(0) + uy'(0) = 0. \]

Since \( T''(u) = T(u)/u^2 - y(0) - uy'(0) \) and \( T'(u) = T(u)/u - uy(0) \), we have

\[ u^2\{T(u)/u^2 - y(0) - uy'(0)\} - 2u\{T(u)/u - uy(0)\} \]

\[ -\{n(n+1) - 2 + \frac{1}{u^2}\}T(u) + y(0) + uy'(0) = 0. \]

Multiplying \( u^2 \) on both sides, we get

\[ \{(n^2 + n - 1)u^2 + 1\}T(u) = (u^4 + u^2)y(0) + (u^3 - u^5)y'(0) \]

for \( E(f(t)) = T(u) \). Hence, we have

\[ T(u) = \frac{u^4 + u^2}{(n^2 + n - 1)u^2 + 1} y(0) + \frac{u^3 - u^5}{(n^2 + n - 1)u^2 + 1} y'(0). \]

Since \( E(t^n) = n! \ u^{n+2} \) and \( y = T^{-1}(u) \), we should obtain the solution of Legendre’s equation in a similar way as Theorem 2.1. Next, let us check the case of the Sumudu representation. Sumudu transform defined by

\[ G(u) = \frac{1}{u} \int_0^\infty e^{-t/u} f(t) dt \]

for \( S[f(t)] = G(u) \). Normally, the slight difference of kernel \( \alpha \cdot e^{-t/u} \) makes many changes for some parameter \( \alpha \), and the difference of the Sumudu transform and Elzaki’s is it. On the other hand, Watugula[24] showed that

\[ S[tf(t)] = u \frac{d}{du} [uG(u)], \]

and Kilicman[6] proved that the Sumudu transform of the \( f^{(n)}(t) \) is a form of

\[ \frac{G(u)}{u^n} - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{u^{n-k}}. \]
for $S[f(t)] = G(u)$ and for $y = f(t)$. Implies,

$$G'(u) = \frac{G(u) - y(0)}{u}$$

and

$$G''(u) = \frac{S[f''(t)] - y''(0)}{u} = \frac{G(u) - y(0)}{u^2} - \frac{y'(0)}{u}$$

are hold. Using this interaction formula, we would like to draw the following theorem.

**Theorem 2.4** Legendre’s equation

$$(1 - t^2)y'' - 2ty' + n(n + 1)y = 0$$

should be expressed by

$$\left\{ \frac{1}{u^2} - 6 + n(n + 1) \right\} G(u) - \left( \frac{1}{u^2} - 6 \right) y(0)$$

$$- \left( \frac{1}{u} - 4u \right) y'(0) + 2u^2 y''(0) = 0$$

by using the Sumudu transform where $n$ is a given constant and $S[f(t)] = G(u)$.

Proof. Let us put $S[f(t)] = G(u)$. Taking the Sumudu transform on both sides of the equation

$$y'' - t^2 y'' - 2ty' + n(n + 1)y = 0,$$

we have

$$\frac{G(u) - y(0)}{u^2} - \frac{y'(0)}{u} - [2G(u) - 2y(0) - 2uy'(0) - 2u^2 y''(0)]$$

$$- 2[2G(u) - 2y(0) - uy'(0)] + n(n + 1)G(u) = 0.$$

Arranging the equation, we have

$$\left\{ \frac{1}{u^2} - 6 + n(n + 1) \right\} G(u) - \left( \frac{1}{u^2} - 6 \right) y(0)$$

$$- \left( \frac{1}{u} - 4u \right) y'(0) + 2u^2 y''(0) = 0$$

for $S[f(t)] = G(u)$. Since $y = G^{-1}(u)$, we can obtain the solution as well.
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Received: December 24, 2913