The Solution of Volterra Integral Equation of the Second Kind by Using the Elzaki Transform

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Abstract

We have checked the Volterra integral equations of the second kind with an integral of the form of a convolution by using the Elzaki transform.

Mathematics Subject Classification: 45B05, 44A05

Keywords: Volterra integral equation, Elzaki transform

1 Introduction

The Volterra integral equations are a special type of integral equations, and they are divided into the first kind and the second[6]. A linear Volterra equation of the second kind has the form of

\[ x(t) = y(t) + \int_a^t k(t, s)x(s)ds, \]
where \( k \) is the kernel. The second (first) kind means that the unknown function \( y \) occurs (does not occur) outside of the integral\cite{7}. In this article, we would like to deal with the form of

\[
x(t) = y(t) + \int_{t_0}^{t} k(t - s)x(s)ds,
\]

where \( k \) is the kernel. On the other hand, the Elzaki transform\cite{2, 4}, a modified Laplace\cite{3}/Sumudu\cite{1, 9, 11} transform, was introduced by Elzaki in 2011 to solve initial value problems in controlling engineering problems, and it is defined by

\[
T(u) = u \int_{0}^{\infty} e^{-t/u}f(t)dt,
\]

for \( E[f(t)] = T(u) \). Elzaki insists that the Elzaki transform should be easily applied to the initial value problems with less computational work, and solve various examples which is not solved by the Laplace or the Sumudu transform\cite{5}. By the reason, we would like to check some Volterra integral equations of the second kind by using the Elzaki transform.

2 The solution of Volterra integral equation of the second kind by using the Elzaki transform

We proved the convolution of Elzaki transform in \cite{10} by the different method with Elzaki. That is

\[
E(f \ast g) = \frac{1}{u} E(f)E(g)
\]

for \( E(f) \) is the Elzaki transform of \( f \). In general, we can find the solution of differential equations with variable coefficients by using the Elzaki transform as following:

**Theorem 2.1** The solution of Volterra integral equation of the second kind

\[
y(t) = x(t) + \int_{a}^{t} K(t - \tau)y(\tau)d\tau
\]

(*)

is expressed by

\[
y(t) = E^{-1}(T(u)) = E^{-1}\left(\frac{uX}{u - K}\right),
\]

where \( K \) is the kernel and \( E[y(t)] = T(u) \).
Proof. Let us consider the general case. Let $E[y(t)] = T(u)$, $E(r) = R$ and $E(q) = Q$. If $f(y(t)) = r(t)$ is given, taking the Elzaki transform on both sides, we have
\[
T(u) = \frac{1}{u} E(r) E(q) = \frac{1}{u} RQ
\]
for $Q$ is the transfer function. Let us take the inverse Elzaki transform on both sides, we get
\[
y = T^{-1}(u) = r * q = T^{-1}\left(\frac{1}{u} RQ\right)
\]
for $*$ is the standard notation of convolution.

With the above idea, let us take the Elzaki transform on the equation (*). Then we have
\[
T(u) = X + E(y * k) = X + \frac{1}{u} T(u)K
\]
for $X = E(x)$ and for $K = E(k)$. Organizing the equality, we have
\[
T(u) = \frac{uX}{u - K}
\]
for the kernel $k$. Therefore we have
\[
y(t) = E^{-1}(T(u)) = E^{-1}\left(\frac{uX}{u - K}\right).
\]

Let us check the following examples.

**Example 2.2** Let us consider $y'' + y = 0$ with the initial condition $y(0) = 0$, $y'(0) = 1$.

**Solution.** We can easily find that the solution is $y = \sin t$. Next, we would like to check this by the integral equation. Since the equation is equivalent to
\[
y(t) = t + \int_0^t (\tau - t)y(\tau)d\tau,
\]
we can rewrite the equation with $y = t + \{y * (-I)\}$ for $*$ is the standard notation of convolution and for $I$ is the identity function. Taking the Elzaki transform on both sides, we have
\[
T(u) = u^3 - \frac{1}{u} T(u)u^3 = u^3 - u^2 T(u)
\]
for $E[y(t)] = T(u)$, because of $E(t) = u^3$. Thus we have
\[
T(u) = \frac{u^3}{1 + u^2}.
\]
As we scan a table of the Elzaki transform[5], we have the solution \( y = \sin t \) because of

\[
E[\sin(at)] = \frac{au^3}{1 + a^2u^2}.
\]

Finally, let us directly approach by the Elzaki transform. Note that \( [y''(t)] = T(u)u^2 - y(0) - uy'(0) \) holds, by the definition. Taking the Elzaki transform on both sides, we have

\[
T(u)u^2 - y(0) - uy'(0) + T(u) = 0
\]

for \( E[y(t)] = T(u) \). Collecting the \( T(u) \)-terms, we have

\[
T(u)(\frac{1}{u^2} + 1) = u
\]

and so, we have the solution

\[
y = \sin t.
\]

Now that let us consider the Volterra integral equation of the second kind.

**Example 2.3** Solve the Volterra integral equation of the second kind

\[
y(t) - \int_0^t y(\tau)\sin(t - \tau)d\tau = t \ [7].
\]

**Solution.** The given equation can be rewritten by

\[
y - y \ast \sin t = t.
\]

Let us write \( E[y(t)] = T(u) \) and apply the convolution theorem. Then we have

\[
T(u) - \frac{1}{u}T(u)u^3 \frac{u^3}{1 + u^2} = u^3.
\]

Organizing the equality, we have

\[
T(u) = u^3(1 + u^2) = u^3 + u^5.
\]

As we scan a table of Elzaki transforms[5], we have the answer

\[
y(t) = t + \frac{t^3}{6}.
\]

This is the same result as that of [7].
Example 2.4 Solve the Volterra integral equation
\[ y(t) - \int_0^t \tau y(t - \tau)d\tau = 1 \]
for \(h\) is a hyperbolic function.

Solution. Writing
\[ y - t * y = 1, \]
we have
\[ T(u) - \frac{1}{u} E(t) T(u) = E(1) \]
for \(E[y(t)] = T(u)\). From the table of Elzaki transform\([5]\), we have
\[ T(u) - \frac{1}{u} u^3 T(u) = u^2. \]
Arranging the equality, we have
\[ T(u) = \frac{u^2}{1 - u^2}. \]
Taking the inverse Elzaki transform, we have the solution
\[ y(t) = \cos ht \]
for \(h\) is a hyperbolic function.

It is well known fact that the first order ODE
\[ \frac{dy}{dx} = f(x, y) \]
with the condition \(y(a) = y_0\) is rewritten to
\[ \phi(x) = y_0 + \int_a^x f(t, \phi(t))dt, \]
where \(f\) is continuous and contains the point \((a, y_0)\). Similarly, an initial value problem
\[ y'' + A(t)y' + B(t)y = 0 \]
with the condition \(y(a) = y_0, y'(a) = y_1\) is rewritten to the Volterra integral equation of the second kind
\[ y(t) = f(t) + \int_a^t k(t, \tau)y(\tau)d\tau, \]
where \(K(t, \tau) = -A(\tau) + (\tau - t)(B(\tau) - A'(\tau))\). And the above \(f(t)\) is continuous on \([a, b]\) and the kernel \(k\) is continuous on the triangular region \(R\) in the \(t\tau\)-plane given by \(a \leq \tau \leq t, a \leq t \leq b\). Then we know that (**) has a unique solution\([8]\) \(y\) on \([a, b]\).
References


Received: December 24, 2013