A Maxwellian Formulation
by Cartan’s Formalism

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Abstract

In this brief review we make some remarks on an interesting mathematical method which is, in our opinion, wrongfully undervalued. In many physics textbooks, the Cartan formalism is not explained and students who specialize in another branch of physics or mathematics, do not know the formal elegance of this approach. The aim of this short note is to stimulate curiosity, and it wants to be a starting point to explore this topic.

Keywords: Maxwell equations, Cartan formalism

1 Introduction

In the ordinary three-dimensional Euclidean space of classical physics, a vector is represented by an arrow and it is written by the well-known vector symbol $\vec{V}$. This approach is intuitive and it is very useful for writing vector identity [1]. For example, to write the Stokes theorem using the components, is much more complex than simply writing $\int \vec{V} \cdot d\vec{T} = \int (\vec{\nabla} \wedge \vec{V}) \cdot d\vec{S}$. After the birth of general relativity, spacetime curvature has made it very difficult to extend this intuitive vector approach. In fact, in curved spaces, displacements do not commute and so we are forced to consider a vector simply as a set of
components. For this reason it is necessary to construct a deeper mathematical object. If we have a scalar function $\phi$ on a differentiable Riemannian manifold and $P$ a point with coordinates $\{x^1, ..., x^n\}$, we can evaluate the derivative of $\phi$ with respect to each of the coordinates [2]. By considering a curve, parametrized with a parameter $\lambda$, the directional derivative of $\phi$ along the curve will be

$$\frac{d\phi}{d\lambda} = \frac{\partial\phi}{\partial x^i} \frac{dx^i}{d\lambda}$$

(1)

Since $\phi$ is completely arbitrary, we can think of the derivatives as a set of $n$ operators, denoted by $\frac{\partial}{\partial x^i}$ while $\frac{dx^i}{d\lambda}$ are the components of the tangent vector. Obviously the sum of the vectors is defined in the following way

$$a^\mu \frac{\partial}{\partial x^\mu} + b^\mu \frac{\partial}{\partial x^\mu} = (a^\mu + b^\mu) \frac{\partial}{\partial x^\mu}$$

(2)

It is easy to verify that Cartan vectors form a vector space called tangent space and it is indicated with $T_P$ [3]. Therefore in Cartan formalism a vector is a linear map from functions to real numbers and when the vector $v$ acts on the function $\phi$, it returns the real number $v(\phi)$. If the vector acts on the sum or product of two functions we have

$$\begin{cases} v(af + bg) = av(f) + bv(g) \\ v(fg) = gv(f) +fv(g) \end{cases}$$

(3)

In this formalism if we have, for example, the relativistic quadrivector energy-impulse $P = (E/c, p_x, p_y, p_z)$ we get [1]

$$P = \frac{E}{c} \frac{\partial}{\partial x^0} + p_x \frac{\partial}{\partial x^1} + p_y \frac{\partial}{\partial x^2} + p_z \frac{\partial}{\partial x^3}$$

(4)

The advantage of Cartan formalism is that the curve and its parameterization determine the vector without explicitly introducing the coordinates.

2 Tensors Field

A 1-form is a linear, real valued function of vectors and that is a 1-form $\tilde{\alpha}$ at the point $P$ takes the vector $\vec{V}$ at $P$ and associates a number $\tilde{\alpha}(\vec{V})$ [4]. Since a vector is a differential operator, the forms associate numbers to differential
operators. It is easy to verify that 1-forms form a vector space, which is called the dual vector space and it is indicated with $T^*_P$ [5]. If the argument of a one-form is one of the basis vectors of the tangent space at the point $P$, we have

$$\tilde{\omega}^\nu \left( \frac{\partial}{\partial x^\mu} \right) = \delta^\nu_\mu$$

(5)

It is easy to verify that $\{\tilde{\omega}^\nu\}$ form a basis for one-forms and we can define the components of a 1-form as we define the components of a vector [6]

$$\tilde{\alpha} = \alpha_\mu \tilde{\omega}^\mu$$

(6)

Therefore we have

$$\tilde{\alpha} (\vec{v}^\nu) = \alpha_\mu \omega^{\mu \nu} \left( \frac{\partial}{\partial x^{\nu}} \right) = \alpha_\mu v^{\nu} \omega^{\mu \nu} \left( \frac{\partial}{\partial x^{\nu}} \right) = \alpha_\mu v^{\nu} \delta^\nu_\mu = \alpha_\mu v^\nu$$

(7)

In components formalism we call it scalar product between 1-forms and vectors. The gradient of a function $f$ is a 1-form defined in the following way

$$\tilde{df} \left( \frac{d}{d\lambda} \right) = df \frac{d}{d\lambda}$$

(8)

and it is the 1-form whose value on an element $\vec{v}^\nu$ is the directional derivative of $f$ along a curve whose tangent is $\vec{v}^\nu$ [2],[7]. If we have $f = x^\mu$ and $d/d\lambda = \partial/\partial x^\nu$, we get

$$\tilde{dx}^\mu \left( \frac{\partial}{\partial x^\nu} \right) = \frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu = \tilde{\omega}^\mu$$

(9)

and therefore it is possible to write the relation (6) as

$$\tilde{\alpha} = \alpha_\mu \tilde{dx}^\mu$$

(10)

The definition of a tensor is a generalization of the definition of one-forms. Indeed a tensor is defined to be a linear, real valued function, which takes as arguments $m$ one-forms and $n$ vectors and associates a number to them. For example if $T$ is a (1,2) tensor, then its value on the 1-form $\tilde{\alpha}$ and the vectors $\vec{u}$ and $\vec{v}$ is $T(\tilde{\alpha}, \vec{u}, \vec{v})$. We have to define the tensor product between vectors and 1-forms.
$$\bar{w} \otimes \hat{\beta} \otimes \tilde{\gamma}(\tilde{\alpha}, \bar{u}, \bar{v}) = \tilde{\alpha}(\bar{w})\tilde{\beta}(\bar{u})\tilde{\gamma}(\bar{v})$$

(11)

and it is possible to write

$$T = T^\sigma_{\mu\nu} \frac{\partial}{\partial x^\sigma} \otimes \tilde{dx}^\mu \otimes \tilde{dx}^\nu$$

(12)

We can see that, in Cartan formalism, everything is a linear operator [1],[8],[9],[10]. Now we define the exterior product between 1-forms which generates a 2-form

$$\tilde{\alpha} \wedge \tilde{\beta} = \tilde{\alpha} \otimes \tilde{\beta} - \tilde{\beta} \otimes \tilde{\alpha}$$

(13)

This product is anticommutative and also we have $\tilde{\alpha} \wedge \tilde{\alpha} = 0$. We can write

$$\tilde{\alpha} \wedge \tilde{\beta} = \frac{1}{2}(\alpha_\mu \beta_\nu - \alpha_\nu \beta_\mu)\tilde{dx}^\mu \wedge \tilde{dx}^\nu$$

(14)

The exterior product of $p$ 1-forms is called $p$-form getting

$$\tilde{\alpha} \wedge \tilde{\beta} \wedge \tilde{\gamma} = \tilde{\alpha} \otimes \tilde{\beta} \otimes \tilde{\gamma} + \tilde{\beta} \otimes \tilde{\gamma} \otimes \tilde{\alpha} + \tilde{\gamma} \otimes \tilde{\alpha} \otimes \tilde{\beta}$$

$$-\tilde{\beta} \otimes \tilde{\alpha} \otimes \tilde{\gamma} - \tilde{\alpha} \otimes \tilde{\gamma} \otimes \tilde{\beta} - \tilde{\gamma} \otimes \tilde{\beta} \otimes \tilde{\alpha}$$

(15)

The exterior derivative is a generalization of the gradient of a function. It is a map from $p$-forms to $(p+1)$-forms. If we have a $p$-form $\alpha = \frac{1}{p!}\alpha_{\mu_1...\mu_p}\tilde{dx}^{\mu_1} \wedge \tilde{dx}^{\mu_2}...$, then

$$\tilde{d}\alpha = \frac{1}{p!}\tilde{d}\alpha_{\mu_1...\mu_p}\tilde{dx}^{\mu_1} \wedge \tilde{dx}^{\mu_2}...$$

(16)

In conclusion, the exterior derivative extends the concept of the differential of a function to differential forms of higher degree.

3 Maxwell Equations

It is well known that Maxwell’s equations can be written in a compact way in 4-dimensional Minkowski spacetime
\[ \partial_\mu F^{\mu\nu} = 4\pi J^\mu \] (17)

\[ \partial^\alpha F^{\mu\nu} + \partial^{\mu} F^{\nu\alpha} + \partial^{\nu} F^{\alpha\mu} = 0 \] (18)

where \( J^\mu = (\rho, j_x, j_y, j_z) \) is a 4-current density and \( F^{\mu\nu} \) is the contravariant Faraday tensor

\[
F^{\mu\nu} = \begin{pmatrix}
0 & -E_x & -E_y & -E_z \\
E_x & 0 & -B_z & B_y \\
E_y & B_z & 0 & -B_x \\
E_z & -B_y & B_x & 0
\end{pmatrix}
\] (19)

By considering the covariant Faraday tensor

\[
F_{\mu\nu} = \begin{pmatrix}
0 & E_x & E_y & E_z \\
-E_x & 0 & B_z & B_y \\
-E_y & B_z & 0 & -B_x \\
-E_z & -B_y & B_x & 0
\end{pmatrix}
\] (20)

We can define the following 2-form

\[
F = \frac{1}{2} F_{\mu\nu} d\tilde{x}^\mu \wedge d\tilde{x}^\nu
\] (21)

that is

\[
F = E_x d\tilde{x}^0 \wedge \tilde{d}x^1 + E_y d\tilde{x}^0 \wedge \tilde{d}x^2 + E_z d\tilde{x}^0 \wedge \tilde{d}x^3 \\
- B_z d\tilde{x}^1 \wedge \tilde{d}x^2 + B_y d\tilde{x}^1 \wedge \tilde{d}x^3 - B_x d\tilde{x}^2 \wedge \tilde{d}x^3
\] (22)

It is easy to verify that, if we impose that the exterior derivative is zero, we get

\[
\left\{ \begin{array}{l}
\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_z}{\partial t} \\
\frac{\partial E_y}{\partial x} - \frac{\partial E_z}{\partial y} = -\frac{\partial B_x}{\partial t} \\
\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0
\end{array} \right. \Rightarrow \left\{ \begin{array}{l}
\nabla \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\
\nabla \cdot \vec{B} = 0
\end{array} \right.
\] (23)
Moreover we can define a 2-form starting from the dual Faraday tensor

\[ F_{\mu\nu}^* = \frac{1}{2} \varepsilon_{\mu\nu\lambda\rho} F^{\lambda\rho} = \begin{pmatrix}
0 & B_x & B_y & B_z \\
-E_x & 0 & E_z & -E_y \\
-B_y & -E_z & 0 & E_x \\
-B_z & E_y & -E_x & 0
\end{pmatrix} \]  

(24)

and we can consider the three-form for the electromagnetic source

\[ *J = \frac{1}{3!} \varepsilon_{\mu\nu\sigma\tau} J^\tau dx^\mu \wedge dx^\nu \wedge dx^\sigma \]  

(25)

where \( \varepsilon_{\mu\nu\sigma\tau} \) is the Levi-Civita symbol. It is easy to verify that

\[
\begin{align*}
\nabla \cdot \vec{E} &= \frac{\rho}{\varepsilon_0} \\
\nabla \wedge \vec{B} &= \mu_0 \vec{J} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \Rightarrow \tilde{d}^* F = 4\pi * J
\end{align*}
\]

(26)

In Cartan formalism Maxwell equations are completely coordinate free and they have this form in any manifold with metric.

4 Conclusion

This brief review attempts to serve as an introduction to Cartan formalism. The advantage of describing vectors via directional derivatives along the curves, is that the curve and its parameterization determine the vector without explicitly introducing the coordinates. From a purely physical point of view all this is not fundamental but from the mathematical point of view it is wonderful. In conclusion we can affirm that God said: ”\( \tilde{d}F = 0 \) and \( \tilde{d}^* F = 4\pi * J \) and then there was light”.

References


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