On Question About Representation of Attraction Sets in an Impulse Control Problem

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Abstract
We study the asymptotic behavior of terminal positions attainability sets in an abstract linear control problem. We assume that energy resources available for the control accumulate over time. A control coefficient in the right-hand part of the differential equation can be a discontinuous function. We assume that admissible controls have a one-impulse nature asymptotically. This motivate us to study attraction sets for suitable characterizations of admissible controls set. For this purpose we used an extension construction in the class of finitely additive measures. The representations of these attraction sets was obtained. The coincidence of the attraction sets was shown.

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We study attraction sets caused by controls with an impulse nature. In connection with the impulse control we note the original approach of N.N. Krasovskii [1] related to the use of generalized functions and approaches [2–5]. We will implement extension constructions, which are widely used in control and game theory problems; see [6,7] and, for example, [8–10]. We consider a linear system

\[ \dot{x}(t) = C(t)x(t) + b(t)u(t); \quad x(0) = x_0 \in \mathbb{R}^k; \quad t \in [0, 1]; \]
here \( u = u(\cdot) \) is a control. Let \( I \triangleq [0,1], I_0 \triangleq [0,1], I_1 \triangleq [0,1[. \) We assume that \( C(t), t \in I \), is a \( k \times k \) matrix and all entries of the matrix function \( C(\cdot) \) are continuous functions on \( I; b = (b_i(\cdot))_{i \in \mathbb{R}^k} \in \{ I_1 \to \mathbb{R}^k \} \). An admissible open-loop control \( u \) is any nonnegative right continuous function from \( \mathbb{R} I_1 \) that satisfies the condition

\[
\int_0^t u(t) \, dt \in [0, r(t)] \quad \forall t \in I_0,
\]

where \( r \) is a continuous nondecreasing function from \( \{ I \to [0, \infty[ \} \). We do not suppose that admissible controls are step functions (in contrast to [4]). Also, (1) is more general impulse constraint than one considered in [4, (2.3)]. The condition is natural for technical systems and means that the energy resources available for the control accumulate over time. Let \( \mathcal{L} \triangleq \{ [a, b]: (a, b) \in I \times I \}; \mathcal{L} \) is a semi algebra of subsets of \( I_1 \). By \( B(I_1, \mathcal{L}) \) we denote the set of all uniform limits of sequences of right continuous step functions from \( \mathbb{R} I_1 \). We assume that \( b_i(\cdot) \in B(I_1, \mathcal{L}) \) \( \forall i \in \mathbb{N} \); this is the key assumption. Let \( \mathcal{F} \) be the set of all admissible controls, i.e., \( \mathcal{F} \) is the set of all nonnegative functions from \( B(I_1, \mathcal{L}) \) that satisfy (1). For any set \( X \) by \( \beta_0[X] \) denote the set of all filter bases on \( X \). If \( \tau \) is a topology on \( X \), then \((\tau - \text{comp})[X] \) is the set of all nonempty compact subsets of \( X \) w.r.t. \( \tau \), and \( \text{cl}(Y, \tau) \) is the closure of \( Y, Y \subset X \). We study the asymptotics of attainability sets for the cases when controls are allowed to be different from zero only on a ‘small’ time period that is less or equal to some predefined value. This leads to the following definitions: \( \text{supp}[w] \triangleq \{ \tau \in I_1 \mid w(\tau) \neq 0 \} \forall w \in \mathcal{F}; \)

\[
F_\kappa \triangleq \{ w \in \mathcal{F} \mid \exists t \in I_1 : \text{supp}[w] \subset [t, t + \kappa[ \} \subset \mathcal{F} \quad \forall \kappa \in ]0, \infty[;
\]

\( \mathcal{F} \triangleq \{ F_\kappa : \kappa \in ]0, \infty[ \}. \) Then \( \mathcal{F} \in \beta_0[\mathcal{F}] \).

We define the terminal position operator \( g \) by the rule

\[
u \mapsto \Phi(1, 0)x_0 + \int_0^1 u(\zeta)\Phi(1, \zeta)b(\zeta) \, d\zeta : \mathcal{F} \to \mathbb{R}^k,
\]

where \( \Phi \) is the principal solution matrix of the homogeneous system \( \dot{x} = C(t)x \).

We will consider \( \mathcal{F} \) as constraints of an asymptotic character (see [11–14]). We define the terminal positions attraction set \([12, (3.3.10), (5.2.24)]:\)

\[
A \triangleq \bigcap_{\varepsilon \in ]0, \infty[} \text{cl}(g^t(F_\varepsilon), \tau^{(k)}_\mathbb{R}) \in (\tau^{(k)}_\mathbb{R} - \text{comp})[\mathbb{R}^k];
\]

where \( \tau^{(k)}_\mathbb{R} \) is the topology of coordinate-wise convergence in \( \mathbb{R}^k \). Let

\[
H_t \triangleq \left\{ (\alpha_1, \alpha_2) \in [0, \infty]^2 \mid (\alpha_1 + \alpha_2) \in [0, r(t)] \right\} \quad \forall t \in I.
\]
We define controls that have no more than two 'steps' on $I_1$. Let $\gamma > 0$, $t \in I_1$, $t_\gamma \overset{\Delta}{=} \inf(\{t + \gamma; 1\})$, $(p, q) \in H_t$,

$$f_{t, \gamma}^{(p, q)}(\tau) \overset{\Delta}{=} \begin{cases} 
\frac{2p}{t_\gamma - t}, & \tau \in \left[t, t + \frac{t_\gamma - t}{2}\right[,
\frac{2q}{t_\gamma - t}, & \tau \in \left[t + \frac{t_\gamma - t}{2}, t_\gamma\right[,
0, & \tau \in I_1 \setminus \left[t, t_\gamma\right[.
\end{cases} \quad (4)$$

Using (4), we define the other set of controls that satisfy (1):

$$F_\kappa^0 \overset{\Delta}{=} \{f_{t, \gamma}^{(p, q)} : t \in I_1, \gamma \in [0, \kappa], (p, q) \in H_t \} \subset F_\kappa \forall \kappa \in [0, \infty[. \quad (5)$$

Let $\mathcal{F}_0 \overset{\Delta}{=} \{F_\kappa^0 : \kappa \in [0, \infty]\}$. Then $\mathcal{F}_0 \in \beta_0[\mathbb{F}]$. We define the attraction set (similarly to (3))

$$A_0 \overset{\Delta}{=} \bigcap_{\epsilon \in [0, \infty]} \text{cl}(g^1(F_\epsilon^0), \tau^{(k)}_{\mathbb{R}}) \in (\tau^{(k)}_{\mathbb{R}} - \text{comp})[\mathbb{R}^k].$$

We note that $A, A_0$ allow sequential representations [12, Proposition 3.3.1]. Indeed, $A_0$ is the set of all $x$ in $\mathbb{R}^k$ such that it is possible to define a sequence $(f_i)_{i \in \mathbb{N}}$ in $\mathbb{F}$ with the following properties:

$$\left( \forall \kappa > 0 \exists m \in \mathbb{N} \forall d \in \overline{m, \infty} : f_d \in F_\kappa^0 \& \big((g(f_i))_{i \in \mathbb{N}} \overset{\tau^{(k)}_{\mathbb{R}}}{\rightarrow} x \right).$$

We will provide an example of such a sequence in the proof of Lemma 2.

We use an extension construction in the class of finitely additive measures; see [4, p.1729], [12, 13]. Let $\lambda$ be the trace of the Lebesgue measure on $\mathcal{L}$ and $(\text{add})_+[\mathcal{L}]$ be the set of all nonnegative finitely additive measures on $\mathcal{L}$. By definition, put $\mathcal{K} \overset{\Delta}{=} \{\mu \in (\text{add})_+[\mathcal{L}] \mid \mu([0, t]) \leq r(t) \forall t \in I_0\}$. We define a mapping $\mathbf{m}$ from $\mathbb{F}$ into $\mathcal{K}$ by the rule

$$\mathbf{m}(f) \overset{\Delta}{=} f * \lambda \forall f \in \mathbb{F}, \quad (6)$$

where $f * \lambda$ stands for an indefinite $\lambda$-integral of $f$. We define $\mathbf{s}$ by the rule

$$\mu \mapsto \Phi(1, 0)x_0 + \int_{I_1} \Phi(1, \zeta)b(\zeta)\mu(d\zeta) : \mathcal{K} \rightarrow \mathbb{R}^k. \quad (7)$$

Combining (2), (6), and (7), we get that $g(u) = \mathbf{s}(\mathbf{m}(u)) \forall u \in \mathbb{F}$.

By $\tau^*_{\mathcal{K}}$ we denote the subspace topology of $\mathcal{K}$ of topological space $(\text{add})_+[\mathcal{L}]$ with *-weak topology. From [14, Corollary 3.1] we obtain that $A = \mathbf{s}^1(M)$, $A_0 = \mathbf{s}^1(M_0)$ where $M \overset{\Delta}{=} \bigcap_{\epsilon \in [0, \infty]} \text{cl}(\mathbf{m}^1(F_\epsilon), \tau^*_{\mathcal{K}})$, \quad $M_0 \overset{\Delta}{=} \bigcap_{\epsilon \in [0, \infty]} \text{cl}(\mathbf{m}^1(F_\epsilon^0), \tau^*_{\mathcal{K}})$.

From (5) we get that

$$M_0 \subset M. \quad (8)$$
We need the following notations from [4]. If \( t \in I_1 \), then by \( \delta_t \) we denote
the trace of the Dirac measure on \( \mathcal{L} \) centered on \( t \). If \( t \in I_0 \), then by \( \delta_t^- \) we
denote the purely finitely additive measure such that \( \int_{I_1} w d \delta_t^- \) is equal to the
left limit of \( w \) at the point \( t \) for all \( w \) in \( B(I_1, \mathcal{L}) \). For details on \( \delta_t, \delta_t^- \) see [4, p. 1735]. By definition, put \( K^* \overset{\triangle}{=} K^*_{(1)} \cup K^*_{(2)} \cup K^*_{(3)} \),
\[
K^*_{(1)} \overset{\triangle}{=} \{ \alpha \delta_0 : \alpha \in [0, r(0)] \}, \quad K^*_{(2)} \overset{\triangle}{=} \{ \alpha \delta_t^- : \alpha \in [0, r(1)] \}, \quad K^*_{(3)} \overset{\triangle}{=} \{ \alpha \delta_t + \alpha \delta_t^- : t \in ]0, 1[, (\alpha_1, \alpha_2) \in H_t \}.
\] (9)

**Lemma 1.** The following inclusion holds \( M \subset K^* \).

The main ideas of the proof are similar to the ones proposed in [4, Lemma 3.1] and [15, Lemma 1].

**Lemma 2.** The following inclusion holds \( K^* \subset M_0 \).

**Proof.** We only sketch the proof for \( K^*_{(3)} \subset M_0 \). The properties \( K^*_{(1)}(\mathcal{L}) \subset M_0, K^*_{(2)}(\mathcal{L}) \subset M_0 \) can be proved in the similar way. Let
\[
T[t] \overset{\triangle}{=} \left\{ \tau \in [0, 1[ \mid \frac{r(t)(\zeta - \tau)}{t - \tau} \in [0, r(\zeta)] \ \forall \zeta \in [\tau, t] \right\} \neq \emptyset \ \forall t \in [0, 1[.
\]
Let \( \eta \) be an arbitrary element of \( K^*_{(3)} \). From (9) we obtain that \( \exists t \in ]0, 1[, (\alpha_1, \alpha_2) \in H_t : \eta = \alpha_1 \delta_t + \alpha_2 \delta_t^- \). To prove that \( \eta \in M_0 \), it is sufficient to construct sequence (see [12, Proposition 3.3.1]) \( f \overset{\triangle}{=} (f_j)_{j \in \mathbb{N}} \) in \( \mathbb{F} \) such that
i) \( \forall \kappa > 0 \ \exists l_* \in \mathbb{N} \ \forall m \in \overrightarrow{l_* \infty} : f_m \in F^0_k \); and ii) \((m(f_j))_{j \in \mathbb{N}} \overset{\tau}{\rightarrow} \eta \).

Let \( t_{\inf} \overset{\triangle}{=} \inf(T[t] \cup \{1 - t\}) \). We introduce sequences: \( t_k^{(1)} \overset{\triangle}{=} t - \frac{t_{\inf}}{2k}, t_k^{(2)} \overset{\triangle}{=} t + \frac{t_{\inf}}{2k}; k \in \mathbb{N} \). We define \( f \) as the sequence of functions \( f_k \in \mathbb{F} \) such that
\[
\left( f_k(\xi) \overset{\triangle}{=} 0 \ \forall \xi \in I_1 \setminus [t_k^{(1)}, t_k^{(2)}] \right) \ \& \ \left( f_k(\xi) \overset{\triangle}{=} c_k^{(1)} \ \forall \xi \in [t_k^{(1)}, t_k^{(2)}] \right) \ \& \ \left( f_k(\xi) \overset{\triangle}{=} c_k^{(2)} \ \forall \xi \in [t_k^{(1)}, t_k^{(2)}] \right), \ k \in \mathbb{N} .\] Obviously, \( f \) fulfills i). The proof of ii) for \( f \) is similar to the proofs of [4, Lemma 3.2] and [15, Lemma 2]. \( \square \)

**Theorem 3.** The following equations hold \( K^* = M = M_0 \).

To prove this theorem we combine (8), Lemma 1, and Lemma 2. The theorem shows a constructive way for the representation of sets \( M = M_0 \). This allows us to construct attraction sets \( A, A_0 \) efficiently. Moreover, the theorem shows insensibility of \( K^* \) to constraints of the asymptotic character within some range (see \( \mathcal{F}, \mathcal{F}_0 \)). Moreover, we obtain that \( A = A_0 \). For attraction sets this shows the asymptotic equivalence of \( \mathcal{F} \) and \( \mathcal{F}_0 \). We can implement this result to study a game problem setting (similar to [15]) with condition (1).

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