Cop-Win Graphs and Robber-Win Graphs

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Abstract

Let $G$ be a finite connected graph of order at least two. Two players (a cop and a robber) are allowed to play a game on $G$ according to the following rule: The cop chooses a vertex to stay then the robber chooses a vertex afterwards. After that they move alternately along edges of $G$, where the robber may opt to stay put when his turn to move comes. The cop wins if he succeeds in putting himself on top of the robber, otherwise, the robber wins. A graph $G$ is said to be a cop-win graph if the cop has a winning strategy on it. Otherwise, $G$ is called a robber-win graph. In this paper we give necessary and sufficient conditions for the join, corona, and lexicographic product of two connected graphs to be cop-win graphs. It is shown that the cartesian product $G \times H$ of any connected graphs $G$ and $H$ of orders at least two is a robber-win graph.

Mathematics Subject Classification: 05C12

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1 Introduction

Let $G$ be a finite connected undirected graph of order at least two. Two players (a cop and a robber) are allowed to play a game on $G$ according to the following rule: The cop chooses a vertex to stay then the robber chooses his afterwards. After that they move alternately along edges of $G$, where the robber may opt...
to stay put when his turn to move comes. The cop wins if he succeeds in putting himself on top of the robber, otherwise, the robber wins. A graph $G$ is said to be a **cop-win graph** if the cop has a winning strategy on it. Otherwise, $G$ is called a **robber-win graph**.

Obviously, given a graph $G$, one of the players must win. This game, also called a game of cops and robbers, was considered by Aigner and Fromme in [1]. They observed that if the cop wins the game on $G$, then $G$ must have a pitfall. A vertex $p$ of $G$ is **pitfall** if there exists a vertex $d$ (called a dominating vertex) of $G$ such that $N_G[p] = N_G(p) \cup \{p\} \subseteq N_G[d]$, where $N_G(p) = \{v \in V(G) : vp \in E(G)\}$. A number of interesting results were obtained by the authors. In particular, they gave a necessary and sufficient condition for a graph to be cop-win graph. For some robber-win graphs, the authors also determined the minimum number of cops (called the cop number of graph $G$) needed to be able to catch the robber in $G$.

The same game has also been studied previously by Nowakowski and Winkler [3], and Quilliot [6]. Other interesting variations of the game were studied by many others (see [4], [5], and [7]).

## 2 Results

The first result is due to Aigner and Fromme [1].

**Theorem 2.1** Graph $G$ is a cop-win graph if and only if by successively removing pitfalls (in any order) $G$ can be reduced to a single vertex.

The above result can be used to prove the next results.

**Theorem 2.2** Let $G$ and $H$ be graphs. Then $G + H$ is a cop-win graph if and only if $G$ or $H$ is a cop-win graph.

**Proof:** Suppose $G$ is a cop-win graph. Then successive removal of pitfalls reduces $G$ to a single vertex, say $v$. Let $G_1, G_2, \ldots, G_k$ be the sequence of subgraphs of $G$ obtained by successively removing pitfalls. Then, for each $i = 1, 2, \ldots, k$, $G_i$ is the subgraph obtained from $G_{i-1}$ by removing from it its pitfalls. Here, $G_0 = G$ and $G_k = \langle v \rangle$. Note that for each stage $i$, every pitfall $p$ of $G_{i-1}$ is a pitfall of $G_{i-1} + H$. Thus removal of these pitfalls (with relative to $G + H$) reduces $G + H$ into the subgraph $K = \langle v \rangle + H$. Clearly, removal of these $n$ vertices of $H$ reduces $K$ to the subgraph $\langle v \rangle$. Therefore, by Theorem 2.1, $G + H$ is a cop-win graph.

Suppose now that $G + H$ is a cop-win graph. Suppose further that $G$ and $H$ are both robber-win graphs. Let the cop choose the vertex $v \in V(G + H)$. Without loss of generality, assume that $v \in V(G)$. Then the robber can choose
a vertex in \( G \) where the cop could not catch him. Since \( G \) is a robber-win graph, the cop will not succeed in putting himself on top of the robber if he stays in \( G \). Now, if the cop traverses through an edge to the graph \( H \), then the robber can also move to graph \( H \) and stay on a vertex of \( H \) where the cop could not catch him. Again, since \( H \) is a robber-win graph, the cop will not succeed in putting himself on top of the robber if he stays in \( H \). By staying on the same graph the cop stays, the robber has always a winning strategy on \( G + H \). This implies that \( G + H \) is a robber-win graph, contrary to our assumption. Therefore, \( G \) or \( H \) must be a cop-win graph. \( \square \)

The corona \( G \circ H \) of two graphs \( G \) and \( H \) is the graph obtained by taking one copy of \( G \) of order \( n \) and \( n \) copies of \( H \), and then joining the \( i^{th} \) vertex of \( G \) to every vertex in the \( i^{th} \) copy of \( H \). For every \( v \in V(G) \), denote by \( H^v \) the copy of \( H \) whose vertices are attached one by one to the vertex \( v \). Subsequently, denote by \( v + H^v \) the subgraph of the corona \( G \circ H \) corresponding to the join \( \langle \{v\} \rangle + H^v, v \in V(G) \).

**Theorem 2.3** Let \( G \) be a connected graph and let \( H \) be any graph. Then \( G \circ H \) is a cop-win graph if and only if \( G \) is a cop-win graph.

**Proof**: For each \( v \in V(G) \), let \( H^v \) be the copy of \( H \) that is attached to the vertex \( v \) in \( G \circ H \). Then every vertex of \( H^v \) is a pitfall of \( G \circ H \). Therefore, removal of these pitfalls (vertices of \( H^v \) for every \( v \in V(G) \)) reduces \( G \circ H \) to the graph \( G \). Thus, by Theorem 2.1, successive removing of pitfalls reduces \( G \circ H \) to a vertex if and only if \( G \) is a cop-win graph. In other words, \( G \circ H \) is a cop-win graph if and only if \( G \) is a cop-win graph. \( \square \)

The lexicographic product \( G[H] \) of two graphs \( G \) and \( H \) is the graph with \( V(G[H]) = V(G) \times V(H) \) and \( (u, u')(v, v') \in E(G[H]) \) if and only if either \( uv \in E(G) \) or \( u = v \) and \( u'v' \in E(H) \).

**Lemma 2.4** Let \( G \) and \( H \) be connected graphs. If \( p \) is a pitfall of \( H \), then \((a, p)\) is a pitfall of \( G[H] \) for every \( a \in V(G) \).

**Proof**: Let \( d \) be a vertex of \( H \) with \( N_H[p] \subseteq N_H[d] \) and let \((b, q) \in N_{G[H]}((a, p))\). Then \( ab \in E(G) \) or \( a = b \) and \( pq \in E(H) \). If \( ab \in E(G) \), then \((b, q) \in N_{G[H]}((a, d))\). Suppose \( a = b \) and \( pq \in E(H) \). Then \( q \in N_H(p) \), so \( q \in N_H[d] \). If \( q = d \), then \((b, q) = (a, d) \). If \( q \neq d \), then \((b, q)(a, d) \in E(G[H])\). Thus \((b, q) \in N_{G[H]}((a, d)) \). Consequently, \( N_{G[H]}((a, p)) \subseteq N_{G[H]}((a, d)) \). Since \((a, p)(a, d) \in E(G[H])\), it follows that \( N_{G[H]}((a, p)) \subseteq N_{G[H]}((a, d)) \), showing that \((a, p)\) is a pitfall in \( G[H] \). \( \square \)

**Theorem 2.5** Let \( G \) and \( H \) be connected graphs. Then \( G[H] \) is a cop-win graph if and only if \( G \) and \( H \) are cop-win graphs.
Proof: Suppose $G$ and $H$ are cop-win graphs. By Theorem 2.1, successive removal of pitfalls reduces $H$ to a single vertex, say $u$. Let $H_1, H_2, \ldots, H_k$ be the sequence of subgraphs of $H$ obtained by successively removing pitfalls, where $H_0 = H$ and $H_k = \langle u \rangle$. By repeatedly applying Lemma 2.4, $G[H]$ can be successively reduced to graphs $G[H_1], G[H_2], \ldots, G[H_k]$. Since $H_k = \langle u \rangle$, it follows that $G[H_k] \cong G$. Since $G$ is a cop-win graph, successive removal of pitfalls reduces it to a graph $\langle v \rangle$. This implies that successive removal of pitfalls reduces $G[H]$ to the graph $\langle (v, u) \rangle$. Therefore $G[H]$ is a cop-win graph.

Suppose $G[H]$ is a cop-win graph. Suppose further that $G$ or $H$, say $G$, is a robber-win graph. Then the robber can stay on a copy of $G$ throughout the game without being caught by the cop. It follows that $G[H]$ is a robber-win graph, contrary to the assumption. □

The Cartesian product $G \Box H$ of two graphs $G$ and $H$ is the graph with $V(G \Box H) = V(G) \times V(H)$ and $(u, u')(v, v') \in E(G \Box H)$ if and only if either $uv \in E(G)$ and $u' = v$ or $u = v$ and $u'v' \in E(H)$.

**Theorem 2.6** Let $G$ and $H$ be connected graphs of orders at least two. Then $G \Box H$ is a robber-win graph.

Proof: It suffices to show that $G \Box H$ has no pitfall. To this end, let $(a, b) \in V(G \Box H)$ and let $(x, y) \in N_{G \Box H}((a, b))$. Consider the following cases:

Case 1. Suppose $x = a$.

Then $by \in E(H)$. Pick $c \in V(G)$ such that $ac \in E(G)$. Then $(a, b)(c, b) \in E(G \Box H)$. Since $b \neq y$, $(x, y)(c, b) \notin E(G \Box H)$. It follows that $N_{G \Box H}((a, b))$ is not contained in $N_{G \Box H}((x, y))$.

Case 2. Suppose $y = b$.

Then $ax \in E(G)$. Pick $z \in V(H)$ such that $yz \in E(H)$. Then $(a, b)(a, z) \in E(G \Box H)$. Since $a \neq x$, $(x, y)(a, z) \notin E(G \Box H)$. It follows that $N_{G \Box H}((a, b))$ is not contained in $N_{G \Box H}((x, y))$.

Since $(x, y)$ is an arbitrary neighbor of $(a, b)$, it follows that $(a, b)$ is not a pitfall of $G \Box H$. Also, since $(a, b)$ was arbitrarily chosen, $G \Box H$ has no pitfall. Accordingly, $G \Box H$ is a robber-win graph. □

References


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