Rw-Connectedness and rw-Sets
in the Product Space

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Abstract

In this paper, the concept of rw-connectedness and rw-sets in the product space is studied. Specifically, this paper characterized rw-connectedness in terms of rw-open and rw-closed sets and rw-continuous functions. This also established some results involving regular open, regular semiopen, rw-interior, and rw-closed sets in the product of subsets of a topological space.

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1 Introduction

In 1937, Stone [6] introduced and investigated the regular open sets. These sets are contained in the family of open sets since a set is regular open if it is equal to the interior of its closure. In 1978, Cameron [2] also introduced and investigated the concept of a regular semiopen set. A set \( A \) is regular semiopen if there is a regular open set \( U \) such that \( U \subseteq A \subseteq \overline{U} \). In 2007, a new class of sets called regular \( w \)-closed sets (\( rw \)-closed sets) was introduced by Benchalli

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and Wali [1]. A set $B$ is $rw$-closed if $\bar{B} \subseteq U$ whenever $B \subseteq U$ for any regular semiopen set $U$. They proved that this new class of sets is properly placed in between the class of $w$-closed sets [5] and the class of regular generalized closed sets [4].

In this paper, the concepts of $rw$-connectedness and $rw$-open sets in the product space are further investigated.

Throughout this paper, space $(X, T)$ (or simply $X$) always means a topological space on which no separation axioms are assumed unless explicitly stated. For a subset $A$ of a space $X$, $\overline{A}$, $\text{int}(A)$, and $C(A)$ denote the closure of $A$, interior of $A$, and complement of $A$ in $X$, respectively.

2 Preliminaries

Definition 2.1 [1] A function $f : X \to Y$ is called

(i) $rw$-open if the image $f(A)$ is $rw$-open in $Y$ for each open set $A$ in $X$.

(ii) $rw$-closed if the image $f(A)$ is $rw$-closed for each closed set $A$ in $X$.

(iii) $rw$-continuous if for every open subset $U$ of $Y$, $f^{-1}(U)$ is $rw$-open in $X$.

(iv) regular strongly continuous (briefly $rs$-continuous) if the inverse image of every $rw$-open set in $Y$ is open in $X$, that is, $f^{-1}(A)$ is open in $X$ for all $rw$-open sets $A$ in $Y$.

3 $rw$-connectedness

Definition 3.1 A space $(X, T)$ is $rw$-connected if it is not the union of two nonempty disjoint $rw$-open sets. Otherwise, a space $(X, T)$ is $rw$-disconnected. A subset $A$ of a topological space is $rw$-connected if it is $rw$-connected as a subspace of $X$.

Remark 3.2 A space $(X, T)$ is $rw$-disconnected if there exist a disjoint nonempty $rw$-open sets $A$ and $B$ such that $X = A \cup B$. The set $A \cup B$ is called the $rw$-decomposition of $X$.

Theorem 3.3 Let $X$ be any space and let $\chi_A : X \to 2$ be the characteristic function of a subset $A$ of $X$. Then $\chi_A$ is $rw$-continuous if and only if $A$ is both $rw$-open and $rw$-closed.

Proof: Suppose that $\chi_A$ is $rw$-continuous. Let $O_1 = \{1\}$ and $O_2 = \{0\}$. Then $O_1$ and $O_2$ are open in $\{0, 1\}$. Since $\chi_A$ is $rw$-continuous, $\chi_A^{-1}(O_1) = A$ and $\chi_A^{-1}(O_2) = C(A)$ are $rw$-open sets in $X$. Thus, $A$ is both $rw$-open and $rw$-closed.
Conversely, let $A$ be both $rw$-open and $rw$-closed in $X$. Let $O$ be an open set in $\{0, 1\}$. Then

$$\chi_A^{-1}(O) = \begin{cases} \emptyset & \text{if } O = \emptyset \\ X & \text{if } O = \{0, 1\} \\ A & \text{if } O = \{1\} \\ C(A) & \text{if } O = \{0\}. \end{cases}$$

It means that $\chi_A^{-1}(O)$ is $rw$-open. Therefore, $\chi_A$ is $rw$-continuous. \qed

**Theorem 3.4** Let $(X, T)$ be a topological space. Then the following statements are equivalent:

(a) $X$ is $rw$-connected.

(b) The only subsets of $X$ both $rw$-open and $rw$-closed are $\emptyset$ and $X$.

(c) No $rw$-continuous function $f : X \to 2$ is surjective, where 2 is the space $\{0, 1\}$ with the discrete topology.

*Proof:* (a) $\Rightarrow$ (b) Let $G$ be both $rw$-open and $rw$-closed set in $X$ and suppose that $G \neq \emptyset, X$. Then $G \cup C(G)$ is an $rw$-decomposition of $X$. It follows that $X$ is not $rw$-connected. Thus, the only subsets of $X$ both $rw$-open and $rw$-closed are $\emptyset$ and $X$.

(b) $\Rightarrow$ (c) Suppose that $f : X \to 2$ is $rw$-continuous and surjective. Then $f^{-1}(\{0\}) \neq \emptyset, X$. Since $\{0\}$ is both open and closed in 2, $f^{-1}(\{0\})$ is both $rw$-open and $rw$-closed. This is a contradiction to our hypothesis. Thus, no $rw$-continuous function $f : X \to 2$ is surjective.

(c) $\Rightarrow$ (a) Suppose that $X$ is $rw$-disconnected. Then $X = A \cup B$, where $A$ and $B$ are disjoint nonempty $rw$-open sets. It follows that $A$ and $B$ are also $rw$-closed sets in $X$. Now, consider the characteristic function $\chi_A$. By Theorem 3.3, $\chi_A$ is $rw$-continuous and surjective. This contradicts our assumption. Therefore, $A$ is $rw$-connected. \qed

**Theorem 3.5** Every $rw$-connected space is connected.

*Proof:* Suppose that a space $X$ is $rw$-connected and $X$ is not connected. Then there exist two nonempty disjoint open sets $O_1$ and $O_2$ such that $X = O_1 \cup O_2$. Thus $X$ is also the union of two nonempty disjoint $rw$-open sets. Thus, $X$ is not $rw$-connected which is a contradiction. Therefore, $X$ is connected. \qed

**Remark 3.6** The converse of Theorem 3.5 is not true.
To see this, consider the space \((X,T)\) where \(X = \{a,b,c\}\) and 
\(T = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}.\) Then the possible decomposition of \(X\) is 
\(\{a,b\} \cup \{c\}\) but \(\{c\}\) is not open. Thus, \(X\) is connected. The rw-open sets 
in \(X\) are \(X, \emptyset, \{a\}, \{b\}, \{c\},\) and \(\{a,b\}\). Now, \(X = \{a,b\} \cup \{c\}\) implying that 
\(X\) is rw-disconnected.

**Theorem 3.7** The rw-continuous image of an rw-connected set is connected.

**Proof:** Let \(X\) be an rw-connected set and let \(f : X \to f(X)\) be an 
rw-continuous function. Suppose that \(f(X)\) is disconnected. Then by there 
exists a continuous surjection \(g : f(X) \to 2.\) Hence, 
g \circ f : X \to 2 is an rw-continuous surjection which is a contradiction to 
Theorem 3.4. Therefore, \(f(X)\) is connected. \(\Box\)

4 **rw-sets in the Product Space**

Throughout this section, let \(\{Y_\alpha| \alpha \in A\}\) be family of topological spaces, 
\(\prod\{Y_\alpha| \alpha \in A\}\) be the cartesian product space, \(A_i\) and \(B_i\) are subsets of \(Y_i.\)

**Theorem 4.1** If \(A\) and \(B\) are subsets of \(X\) with \(A \subseteq B,\) then rw-(\(A\)) \(\subseteq\) rw-(\(B\)).

**Lemma 4.2** \(\prod_{i=1}^{n} B_i\) is regular open if and only if \(B_i\) is regular open for every 
\(i = 1, 2, ..., n.\)

**Proof:** Let \(\prod_{i=1}^{n} B_i\) be a regular open set. Then

\[
\text{int}\left(\prod_{i=1}^{n} B_i\right) = \text{int}\left(\prod_{i=1}^{n} B_i\right) = \prod_{i=1}^{n} \text{int}(B_i) = \prod_{i=1}^{n} B_i.
\]

Therefore, \(\text{int}(B_i) = B_i.\) Hence, \(B_i\) is regular open. 
The converse is proved similarly. \(\Box\)

**Lemma 4.3** If \(A_i\) is regular semiopen for every \(i = 1, 2, ..., n,\) then \(\prod_{i=1}^{n} A_i\) is 
regular semiopen.
Proof: Let $A_i$ be regular semiopen for every $i = 1, 2, ..., n$. Then there exists a regular open $U_i$ such that $U_i \subseteq A_i \subseteq \overline{U_i}$. By Theorem 4.2, $\prod_{i=1}^{n} U_i$ is regular open and

$$\prod_{i=1}^{n} U_i \subseteq \prod_{i=1}^{n} A_i \subseteq \prod_{i=1}^{n} U_i = \prod_{i=1}^{n} U_i.$$ Therefore $\prod_{i=1}^{n} A_i$ is regular semiopen. \hfill \Box

Remark 4.4 If $A$ is regular open (regular semiopen) in $\prod_{i=1}^{n} Y_i$, then $A$ is not necessarily a cartesian product of regular open (regular semiopen) sets in $Y_i$.

Lemma 4.5 If $\prod_{i=1}^{n} F_i$ is rw-closed in $\prod_{i=1}^{n} X_i$, then $F_i$ is rw-closed in $X_i$ for every $i = 1, 2, ..., n$.

Proof: Suppose that $\prod_{i=1}^{n} F_i$ is rw-closed in $\prod_{i=1}^{n} X_i$ and let $F_i \subseteq U_i$ where $U_i$ is regular semiopen. Then $\prod_{i=1}^{n} F_i \subseteq \prod_{i=1}^{n} U_i$. Since $\prod_{i=1}^{n} F_i$ is rw-closed and $\prod_{i=1}^{n} U_i$ is regular semiopen by Lemma 4.3, $\prod_{i=1}^{n} F_i \subseteq \prod_{i=1}^{n} U_i$. But $\prod_{i=1}^{n} F_i = \prod_{i=1}^{n} F_i \subseteq \prod_{i=1}^{n} U_i$ implies that $\overline{F_i} \subseteq U_i$ for every $i = 1, 2, ..., n$. Therefore, $F_i$ is rw-closed for every $i = 1, 2, ..., n$. \hfill \Box

Lemma 4.6 $rw\text{-}int}(A) = C(rw\text{-}(C(A))).$

Proof:

$$x \in rw\text{-}int(A) \iff x \in O \text{ for some } rw\text{-}open \text{ set } O \text{ with } O \subseteq A \iff x \notin C(O) \text{ for some } rw\text{-}closed \text{ set } C(O) \text{ with } C(A) \subseteq C(O) \iff x \notin rw\text{-}(C(A)) \iff x \in C(rw\text{-}(C(A))).$$

This completes the proof. \hfill \Box

Lemma 4.7 $rw\text{-}int}(A) = C(rw\text{-}(C(A))).$
Proof:

\[ x \in \text{rw-int}(A) \iff x \in O \text{ for some } \text{rw-open set } O \text{ with } O \subseteq A \]
\[ \iff x \notin C(O) \text{ for some } \text{rw-closed set } C(O) \]
\[ \quad \text{with } C(A) \subseteq C(O) \]
\[ \iff x \notin \text{rw-}(C(A)) \]
\[ \iff x \in C(\text{rw-}(C(A))) \]

This completes the proof. \(\square\)

**Theorem 4.8** \(\text{rw-int} \left( \prod_{i=1}^{n} A_i \right) = \prod_{i=1}^{n} \text{rw-int}(A_i)\).

**Proof:** By Lemma 4.7, and Theorem 4.1,

\[
\text{rw-int} \left( \prod_{i=1}^{n} A_i \right) = C \left( \text{rw-} \left( C(\prod_{i=1}^{n} A_i) \right) \right) \\
= C \left( \text{rw-} \left( \bigcup_{i=1}^{n} (C(A_i)) \right) \right) \\
= C \left( \bigcup_{i=1}^{n} \text{rw-}(C(A_i)) \right) \\
= \bigcap_{i=1}^{n} C(\text{rw-}(C(A_i))) \\
= \bigcap_{i=1}^{n} \langle C(\text{rw-}(C(A_i))) \rangle \\
= \bigcap_{i=1}^{n} \langle \text{rw-int}(A_i) \rangle \\
= \prod_{i=1}^{n} \text{rw-int}(A_i). \quad \square
\]

**References**


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