The Cover Pebbling Number of the Join of Some Graphs

Michael E. Subido and Imelda S. Aniversario

Department of Mathematics and Statistics
College of Science and Mathematics
MSU-Iligan Institute of Technology
Tibanga, Iligan City, Philippines

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Abstract

Given a graph $G$ and a configuration $C$ of pebbles on the vertices of $G$, a pebbling step or move $[u, v]$ consists of removing two pebbles off of one vertex $u$, and then placing one pebble on an adjacent vertex $v$. In a pebbling step $[u, v]$, $u$ is the support vertex while $v$ is a target vertex. A graph is said to be cover-pebbled if every vertex has a pebble on it after a series of pebbling steps. The cover pebbling number $\gamma(G)$ of a graph $G$ is the minimum number of pebbles such that however the pebbles are initially placed on the vertices of $G$ we can eventually put a pebble on every vertex simultaneously by a pebbling step.

In this paper, the cover pebbling number of graphs resulting from the join of two graphs $G$ and $H$ are determined via a key vertex of the graph. In particular, this paper determines the cover pebbling number of the wheels $W_n$, the fans $F_n$, and the join of any graph $G$ with $P_n$ and $C_n$, respectively.
1 Introduction

One recent development in graph theory is the game of pebbling which was first suggested by Lagarias and Saks[6],[8]. It has been the subject of much research and substantive generalizations. It was first introduced by Chung[2], and has been developed by many other graph theorists including Hulbert[4]. Given a connected graph $G$, a specified number of pebbles, and a configuration of the pebbles on the vertices of $G$, the goal is to be able to move at least one pebble to any specified target vertex using a sequence of pebbling moves, where a pebbling move $[u, v]$ consists of removing two pebbles off of one vertex $u$, and then placing one pebble on an adjacent vertex $v$. But how does the pebbling problem change if instead of having a specified target vertex we need to place a pebble simultaneously on every vertex of the graph? In some scenarios this seems to be a more natural question, for example, if information needs to be transmitted to several locations of a network, or if army troops need to be deployed simultaneously. Hence, the most important reachability questions concern the cover pebbling number of a graph.

In 2010, Subido and Aniversario [10] expounded the work of B. Crull, et. al. [3] by finding the cover pebbling number of some common graphs via a key vertex of the graph. They found out that if we can determine the key vertex of any graph $G$, then it can be used to find the cover pebbling number of $G$ in a simpler and more convenient way. Thus in this paper, the cover pebbling number of graphs resulting from the join of two graphs $G$ and $H$ are determined via a key vertex of the graph. In particular, this paper determines the cover pebbling number of the wheels $W_n$, the fans $F_n$, and the join of any graph $G$ with $P_n$ and $C_n$, respectively.

2 Preliminary Concepts and Results

Definition 2.1 A graph $G$ consists of a finite nonempty set $V = V(G)$ together with a set $E = E(G)$ of unordered pairs of distinct elements of $V$. Each element of $V$ is called a vertex of $G$. The sets $V(G)$ and $E(G)$ are called the vertex set and edge set of $G$, respectively. A walk in a graph $G$ is an alternating sequence $W : v_0, e_1, v_1, e_2, \ldots, v_{n-1}, e_n, v_n \ (n \geq 0)$ of vertices and edges beginning and ending with vertices such that every $e_i = v_{i-1}v_i$ is an edge of $G$, $1 \leq i \leq n$. A walk is a path if all its vertices are distinct. A closed walk is called a cycle if its $n$ vertices are distinct and $n \geq 3$. A graph is called a complete graph if every pair of its vertices are adjacent.

Definition 2.2 The join $G + H$ of two graphs $G$ and $H$ is the graph with

$$V(G + H) = V(G) \cup V(H)$$
and

\[ E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G) \text{ and } v \in V(H)\}. \]

**Definition 2.3** The fan \( F_n \) of order \( n + 1 \) is the graph \( P_n + K_1 \). That is, the fan graph is composed of a path consisting of \( n \) vertices, \( v_1, v_2, \ldots, v_n \), which are all connected to a single vertex \( v_0 \), common to all these \( n \) vertices, called the **hub vertex**, for a total of \( n' = n + 1 \) vertices.

**Definition 2.4** The wheel \( W_n \) of order \( n + 1 \) is the graph \( C_n + K_1 \). That is, the wheel graph is composed of a cycle consisting of \( n \) vertices, \( v_1, v_2, \ldots, v_n \), which are all connected to a single vertex \( v_0 \), common to all these \( n \) vertices, called the **hub vertex**, for a total of \( n' = n + 1 \) vertices.

**Definition 2.5** A pebbling of a connected graph \( G \) is a placement of pebbles on the vertices of \( G \).

**Definition 2.6** Suppose \( t \) pebbles are distributed onto the vertices of a graph \( G \). A **pebbling step (move)** \([u, v]\) consists of removing two pebbles off of one vertex \( u \), and then placing one pebble on an adjacent vertex \( v \).

**Definition 2.7** A pebble can be **moved** to a vertex \( v \), called the **target vertex**, if we can repeatedly apply pebbling steps so that in the resulting configuration \( v \) has one pebble.

**Definition 2.8** The **distance** \( \text{dist}(u, v) \) between two vertices \( u \) and \( v \) of a connected graph \( G \) is the length of a shortest path joining \( u \) and \( v \).

**Definition 2.9** The **distance** \( \text{dist}(v) \) of a vertex \( v \) in a graph \( G \) is the sum of the distances from \( v \) to each vertex of \( V(G) \), that is

\[ \text{dist}(v) = \sum_{u \in V(G)} \text{dist}(u, v) \]

for all \( u \in V(G), u \neq v \).

**Definition 2.10** Let \( v \in V(G) \). Then \( v \) is called a **key vertex** if \( \text{dist}(v) \) is a maximum.

**Definition 2.11** The **pebbling number** \( f(G) \) of a graph \( G \) is the least \( t \) such that, however \( t \) pebbles are placed on the vertices of \( G \), we can move a pebble to any vertex by a sequence of pebbling steps.
Definition 2.12 The cover pebbling number \(\gamma(G)\) of a graph \(G\) is the minimum number of pebbles such that however the pebbles are initially placed on the vertices of \(G\) we can eventually put a pebble on every vertex simultaneously by a pebbling move.

Theorem 2.13 The Stacking Theorem [4]
The initial configuration of pebbles that requires the most pebbles to be cover solved happens when all pebbles are placed on a single vertex \(v\), choosing \(v\) such that \(\text{dist}(v)\) is a maximum. That is,
\[
s(v) = \sum_{u \in V(G)} 2^\text{dist}(u,v).
\]
Do this for every vertex \(v \in V(G)\). Then \(\gamma(G)\) is the largest \(s(v)\).

Theorem 2.14 [4] Given a tree \(T\), let \(C\) be a non-coverable simple configuration of maximum size with key vertex \(v\). Then \(v\) is an end of the longest path in \(T\).

Theorem 2.14 simply states that the key vertex \(v\) must be an end vertex of a longest path in \(T\). Hence, for a path \(P_n\) with vertices \(v_1, v_2, \ldots, v_n\), the key vertex is either \(v_1\) or \(v_n\).

Theorem 2.15 [3] Let \(P_n\) be a path. Then \(\gamma(P_n) = 2^n - 1\) for \(n \geq 1\).

Theorem 2.16 [3] Let \(C_n\) be a cycle of order \(n\). Then
\[
\gamma(C_n) = \begin{cases} 
3(2^{\frac{n}{2}} - 1) & \text{if } n \text{ is even}, \\
2^{\frac{n+3}{2}} - 3 & \text{if } n \text{ is odd}
\end{cases}
\]

Theorem 2.17 [3] Let \(K_n\) be a complete graph. Then \(\gamma(K_n) = 2n - 1\) for \(n \geq 1\).

3 Main Results

This section determines the cover pebbling number of the join of two graphs \(G\) and \(H\). In particular, this section determines the cover pebbling number of the wheels \(W_n\) and the fans \(F_n\).
3.1 Cover Pebbling Number of Wheels and Fans

**Theorem 3.1** Let $W_n$ be a wheel of order $n' = n + 1$. For $n \geq 3$, $\gamma(W_n) = 4n - 5 = 4n' - 9$.

*Proof:* Consider a configuration of pebbles in which all the pebbles are on one vertex of $W_n$, say $v$, that is not the hub vertex $v_0$ since $\text{dist}(v_0)$ is not a maximum. Choose $v$ as the key vertex of $W_n$. Then there are three vertices that are at distance 1 from $v$ and $n - 3$ vertices that are at distance 2 from $v$. Hence, $2^1(3)$ pebbles are required to cover the three vertices at distance 1 from $v$ and $2^2(n - 3)$ pebbles are required to cover the remaining $n - 3$ vertices that are at distance 2 from $v$. Thus, $s(v) = 2^0 + 2^1(3) + 2^2(n - 3) = 1 + 2(3) + 4(n - 3) = 1 + 6 + 4n - 12 = 4n - 5$. By Theorem 2.13, we have $\gamma(W_n) = 4n - 5 = 4n' - 9$ since $n' = n + 1$. 

**Theorem 3.2** Let $F_n$ be a fan of order $n' = n + 1$. For $n \geq 3$, $\gamma(F_n) = 4n - 3 = 4n' - 7$.

*Proof:* Let $C$ be a configuration of pebbles in which all the pebbles are placed on the key vertex of $F_n$, say $v$, which is not the hub vertex $v_0$. By definition $F_n \cong K_1 + P_n$ where $K_1 = v_0$. Since the hub vertex is not the key vertex, then it must be on the path $P_n$. By Theorem 2.14, the key vertex must be $v_1$ or $v_n$. Without loss of generality, let $v_1$ be our key vertex. Then there are two vertices at distance 1 from $v_1$ and there are $n - 2$ vertices at distance 2 from $v_1$. For the two vertices at distance 1 from $v_1$, $2^1(2)$ pebbles are required to cover it and for the $n - 2$ vertices at distance 2 from $v_1$, $2^2(n - 2)$ pebbles are required to cover it. Thus, $s(v_1) = 2^0 + 2^1(2) + 2^2(n - 2) = 1 + 2(2) + 4(n - 2) = 1 + 4 + 4n - 8 = 4n - 3$ pebbles are required to cover-solve these vertices. By Theorem 2.13, it follows that $\gamma(F_n) = 4n - 3 = 4n' - 7$ since $n' = n + 1$. 

3.2 Cover Pebbling Number of the Join of Some Families of Graphs

Recall that the join of two graphs $G$ and $H$ denoted by $G + H$ is the graph with $V(G + H) = V(G) \cup V(H)$ and

$$E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G) \text{ and } v \in V(H)\}.$$ 

Among all graphs on $n$ vertices, the complete graph has the smallest cover pebbling number and the path has the largest cover pebbling number, that is, $\gamma(P_n) > \gamma(C_n) > \gamma(K_n)$. This section determines the cover pebbling number of the join of two graphs $G$ and $H$. In particular, this section determines the cover pebbling number of the join of any graph $G$ with $P_n$ and $C_n$, respectively.

**Remark 3.3** The Two-Path Closure Property:

Let $G + H$ be the join of two graphs $G$ and $H$. Then the distance between any two adjacent vertices in $G + H$ is 1, otherwise, the distance is 2.
Remark 3.3 simply states that for the graph \(G + H\), the distance between any two vertices is either 1 or 2. That is, with \(v\) as the key vertex of \(G + H\), the distance between any vertices adjacent to \(v\) is 1 and the distance between any vertices not adjacent to \(v\) is 2.

**Theorem 3.4** Let \(P_n\) be a path of order \(n\) and \(G\) be any graph of order \(m\) such that \(n \geq m\). Then \(\gamma(P_n + G) = 4n + 2m - 5\).

*Proof:* Let \(P_n\) be a path of order \(n\) and \(G\) be any graph of order \(m\) such that \(n \geq m\), labeled sequentially. Since \(n \geq m\) and suppose that a simple configuration of pebbles is placed on the key vertex of \(P_n + G\) such that this key vertex lies on the path \(P_n\). Furthermore, let this configuration of pebbles be placed on \(v_1\) by Theorem 2.14. For the graph \(P_n + G\), there are \(m + 1\) vertices that are adjacent to \(v_1\) and \(n - 2\) vertices that are not adjacent to \(v_1\). By Remark 3.3, there are \(m + 1\) vertices that are at distance 1 from \(v_1\) and there are \(n - 2\) vertices that are at distance 2 from \(v_1\). For the vertices at distance 1 from \(v_1\), \(2^1(m + 1)\) pebbles are required to cover it and for the vertices at distance 2 from \(v_1\), \(2^2(n - 2)\) pebbles are required to cover it. Thus, \(s(v) = 2^0 + 2^1(m + 1) + 2^2(n - 2) = 1 + 2(m + 1) + 4(n - 2) = 1 + 2m + 2 + 4n - 8 = 4n + 2m - 5\) pebbles are required to cover-solve these vertices. Therefore, \(\gamma(P_n + G) = 4n + 2m - 5\) by Theorem 2.13. \(\blacksquare\)

**Example 3.5** Consider the fan \(F_7\) below.

By Theorem 3.2, we have \(\gamma(F_7) = 4(7) - 3 = 28 - 3 = 25\). By definition, \(F_7 \cong P_7 + K_1\). By Theorem 3.4, we have \(\gamma(P_7 + K_1) = 4(7) + 2(1) - 5 = 28 + 2 - 5 = 25\). Therefore, \(\gamma(F_7) = 25 = \gamma(P_7 + K_1)\).

**Theorem 3.6** Let \(C_n\) be a cycle of order \(n\) and \(G\) be any graph of order \(m\) such that \(n \geq m\). Then \(\gamma(C_n + G) = 4n + 2m - 7\).

*Proof:* Let \(C_n\) be a cycle of order \(n\) and \(G\) be any graph of order \(m\) such that \(n \geq m\), labeled sequentially. Suppose that a simple configuration of pebbles
and with vertex sets \( \{24 + 2 \} \). By Theorem 3.1, we have \( \gamma(C_n + G) \). Let \( \gamma \) be our key vertex. For the graph \( C_n + G \), there are \( m + 2 \) vertices that are adjacent to \( v \) and there are \( n - 3 \) vertices that are not adjacent to \( v \). By Remark 3.3, there are \( m + 2 \) vertices that are a distance 1 from \( v \) and there are \( n - 3 \) vertices that are a distance 2 from \( v \). For the vertices at distance 1 from \( v \), \( 2^1(m + 2) \) pebbles are required to cover it and for the vertices at distance 2 from \( v \), \( 2^2(n - 3) \) pebbles are required to cover it. Thus, \( s(v) = 2^0 + 2^1(m + 2) + 2^2(n - 3) = 1 + 2(m + 2) + 4(n - 3) = 1 + 2m + 4 + 4n - 12 = 4n + 2m - 7 \) pebbles are required to cover-solve these vertices. Therefore, \( \gamma(C_n + G) = 4n + 2m - 7 \) by Theorem 2.13.

**Example 3.7** Consider the wheel \( W_6 \) below.

![Figure 2: A wheel of order 6 + 1 = 7](image)

By Theorem 3.1, we have \( \gamma(W_6) = 4(6) - 5 = 24 - 5 = 19 \). By definition, \( W_6 \cong C_6 + K_1 \). By Theorem 3.6, we have \( \gamma(C_6 + K_1) = 4(6) + 2(1) - 7 = 24 + 2 - 7 = 19 \). Therefore, \( \gamma(W_6) = 19 = \gamma(C_6 + K_1) \).

**Theorem 3.8** Let \( K_n \) and \( K_m \) be complete graphs of orders \( n \) and \( m \), \( n \geq m \), and with vertex sets \( \{v_1, v_2, \ldots, v_n\} \) and \( \{u_1, u_2, \ldots, u_m\} \), respectively. Then \( \gamma(K_n + K_m) = 2n + 2m - 1 \).

**Proof:** Let \( K_n \) and \( K_m \) be complete graphs of orders \( n \) and \( m \) and with vertex sets \( \{v_1, v_2, \ldots, v_n\} \) and \( \{u_1, u_2, \ldots, u_m\} \), respectively. Since \( n \geq m \), suppose further that a simple configuration of pebbles is placed on the key vertex of \( K_n + K_m \) such that this key vertex lies on the vertex set of \( K_n \). By Theorem 2.14, let \( v_1 \) be our key vertex. For the graph \( K_n + K_m \), there are \( m + n - 1 \) vertices that are adjacent to \( v_1 \). By Remark 3.3, there are \( m + n - 1 \) vertices that are at distance 1 from \( v_1 \). And for these vertices at distance 1 from \( v_1 \), \( 2^1(m + n - 1) \) pebbles are required to cover it. Thus, \( s(v_1) = 2^0 + 2^1(m + n - 1) = 1 + 2(m + n - 1) = 1 + 2m + 2n - 2 = 2n + 2m - 1 \) pebbles are required to cover-solve these vertices. Therefore, \( \gamma(K_n + K_m) = 2n + 2m - 1 \) by Theorem 2.13.
As a consequence to Theorem 3.8, we have the following corollary.

**Corollary 3.9** Let $G$ and $H$ be graphs of orders $n$ and $m$, respectively, with $n \geq m$. If $G = K_n$ and $H = K_m$, then $\gamma(G + H) = \gamma(G) + \gamma(H) + 1$.

**Proof:** Suppose that $G = K_n$ and $H = K_m$. Then by Theorem 2.15, we have $\gamma(K_n) = 2n - 1$ and $\gamma(K_m) = 2m - 1$, respectively. Therefore,

\[
\gamma(G + H) = \gamma(K_n + K_m)
\]

\[
= 2n + 2m - 1 \quad \text{by Theorem 3.8}
\]

\[
= (2n - 1) + (2m - 1) + 1
\]

\[
= \gamma(K_n) + \gamma(K_m) + 1
\]

\[
= \gamma(G) + \gamma(H) + 1. \quad \blacksquare
\]

4 Recommendation

This paper shows that if we can determine the key vertex of any graph $G$, then it can be used to find the cover pebbling number of $G$ in a simpler and more convenient way. Hence, it is recommended to determine the cover pebbling number of graphs resulting from other graph binary operations such as the Cartesian product, the composition and the corona of two graphs $G$ and $H$, respectively.

References


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