On the Fourier Transform for Heat Equation

P. Haarsa\(^1\) and S. Pothat\(^2\)

\(^1\)Department of Mathematics, Srinakharinwirot
Bangkok 10110, Thailand

\(^2\)Wad Ban-Koh School, Bandara, Amphoe Pichai
Uttaradit 53220, Thailand

Copyright © 2014 P. Haarsa and S. Pothat. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

In this paper, we show that the solution of heat equation can be obtained by using the Fourier transform, the convolution, and the Fourier inversion. The solution we obtained is unique.

Mathematics Subject Classification: 34B05, 35J05, 35J25, 35L05

Keywords: Heat equation, Fourier transform, Fourier inversion

1 Introduction

The heat equation [2] is of fundamental importance in diverse scientific fields. In mathematics, it is the prototypical parabolic partial differential equation. In probability theory, the heat equation is connected with the study of Brownian motion via the Fokker-Planck equation. In financial mathematics it is used to solve the Black-Scholes partial differential equation. The diffusion equation, a more general version of the heat equation, arises in connection with the study of chemical diffusion and other related processes. The heat equation is used in probability and describes random walks. Lunnaree and Nonlaopon [1] present the fundamental solution of operator \((\boxplus + m^2)^k\) and this fundamental solution is called the diamond Klein-Gordon kernel. They also study the Fourier transform of the diamond Klein-Gordon kernel and the Fourier transform of their
convolution. Romero and Cerutti [3] introduce a new definition of the Fractional Fourier transform of order $\alpha$, $0 < \alpha \leq 1$, of a function which belongs to the Lizorkin space of functions. In this paper, we show that the solution of the heat equation can be obtained by using the Fourier transform, the convolution and Fourier inversion theorem. The solution we obtained is unique.

2 Preliminaries

We consider the heat equation with the initial condition in our problem. The Fourier transform and the convolution are used to solve the problem.

Definition 2.1. Fourier transform [4]. Consider a function $f(x)$ defined on the interval $(-\infty, \infty)$. The Fourier transform of $f$ is denoted by the integral

$$F[f](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-i\gamma t}dt.$$  \hspace{1cm} (1)

for $x \in \mathbb{R}$ and the integral exists.

Definition 2.2. Fourier transform of a convolution [4]. The convolution of $f$ and $g$ is the function $(f * g)$ defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y - x)g(y)dy.$$  \hspace{1cm} (2)

It also can be written as $(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x - y)dy$, by the change of variable.

Lemma 2.3. Let $f = f(x)$ be an absolutely integrable function on $(-\infty, \infty)$, and let $\nu$ be a real constant, and define the translate of $f$ by $\nu$ according to the formula $f_\nu(x) = f(x - \nu)$ for all $-\infty < x < \infty$. The Fourier transform of $f_\nu$ satisfies the relation $F(f_\nu)(\gamma) = e^{-i\gamma \nu}F(f)(\gamma)$ for all $-\infty < x < \infty$.

Proof. By definition 2.1., we have derived the Fourier transfer to $f_\nu(x)$ as

$$F(f_\nu)(\gamma) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_\nu(x)e^{-i\gamma x}dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - \nu)e^{-i\gamma x}dx.$$  \hspace{1cm} (3)
Let \( y = x - \nu \). Then, \( dy = dx \). The equation (3) becomes

\[
F(f_{\nu})(\gamma) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)e^{-i\gamma(y+\nu)} dy
\]
\[
= \frac{e^{-i\gamma\nu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)e^{-i\gamma y} dy
\]
\[
= e^{-i\gamma\nu} F(f)(\gamma).
\]

(4)

3 Main Result

We next will show that

\[
\int_{-\infty}^{\infty} e^{-\frac{(x-t-y)^2}{4t}} \chi(y) dy
\]

is a solution of \( u_t - u_{xx} + u_x = 0 \) in \(-\infty < x < \infty, 0 < t < \infty\), that can be derived by using the Fourier transform method subjected to the initial condition \( u(x,0) = \chi(x) \) for all \(-\infty < x < \infty\).

Proof. By taking a Fourier transform to both sides of our given equation and Lemma 2.3., then we obtained

\[
\frac{\partial}{\partial t} F(u)(\gamma) + (i\gamma + \gamma^2)F(u)(\gamma) = 0
\]
\[
\frac{\partial}{\partial t} \left( F(u)(\gamma)e^{(i\gamma+\gamma^2)t} \right) = 0
\]
\[
F(u)(\gamma) = \nu(\gamma)e^{-\gamma^2 t - i\gamma t}.
\]

(5)

Therefore, \( F(\chi)(\gamma) = F(u(.,0))(\gamma) = \nu(\gamma) \) and \( F(u)(\gamma) = F(\chi)(\gamma)e^{-\gamma^2 t - i\gamma t} \).

By table of Fourier transform [4], it implies that \( F(e^{-a(*)^2})(\gamma) = \frac{e^{-\frac{\gamma^2}{2a}}}{\sqrt{2\pi a}} \). Take \( a = \frac{1}{4t} \), and the above equation can be rearranged as \( F\left(\frac{e^{-\frac{\gamma^2}{2t}}}{\sqrt{2t}}\right)(\gamma) = e^{-\gamma^2 t} \).

Therefore,

\[
F(u)(\gamma) = F(\chi)(\gamma)F\left(\frac{e^{-\frac{\gamma^2}{4t}}}{\sqrt{2t}}\right)(\gamma)e^{-i\gamma t}.
\]

(6)

where \( F(f * g)(\gamma) = \sqrt{2\pi}F(f)(\gamma)F(g)(\gamma) \) is a Fourier transform of the convo-
solution \[4\] \((f * g)(x)\). So, the equation (6) becomes
\[
F(u)(\gamma) = \frac{1}{\sqrt{2\pi}} F\left(\chi \ast \frac{e^{-\frac{(\cdot)^2}{4t}}}{\sqrt{2t}}\right)(\gamma) e^{-i\gamma t}
\]
\[
= F\left(\chi \ast \frac{1}{\sqrt{\varphi t}} e^{-\frac{(\cdot)^2}{4t}}\right)(\gamma) e^{-i\gamma t}
\]
\[
= F\left[\left(\chi \ast \frac{1}{\sqrt{\varphi t}} e^{-\frac{(\cdot)^2}{4t}}\right)\right](\gamma).
\] (7)

Equation (7) can be derived from applying Lemma 2.3. by letting \(\nu = t\).

Consequently, the inversion theorem \[4\] implies
\[
u(x,t) = \left(\chi \ast 1 \sqrt{\varphi t} e^{-\frac{(\cdot)^2}{4t}}\right)(x, t)
\]
for all \(-\infty < x < \infty\) and \(0 < t < \infty\).

As a result, the solution of the heat equation for any initial data \(\chi\) is
\[
u(x,t) = \left(\chi \ast 1 \sqrt{\varphi t} e^{-\frac{(\cdot)^2}{4t}}\right)(x, t)
\]
\[
= \int_{-\infty}^{\infty} e^{-\frac{(x,y)^2}{4t}} \chi(y) dy
\]
for all \(-\infty < x < \infty\) and \(0 < t < \infty\).

Next, we show that our solution is the only one solution by the energy method \[4\]. Let \(\Lambda = u_1 - u_2\) Then, it can be rewritten as.
\[
0 = 0.\Lambda = (\Lambda_t - c\Lambda_{xx})(\Lambda) = \left(\frac{1}{2}\Lambda^2\right)_t + (-c\Lambda_x\Lambda)_x + c\Lambda_x^2.
\] (8)

By integrating over the interval \(0 < x < l\), we obtain
\[
0 = \int_0^l \left(\frac{1}{2}\Lambda^2\right) dx - c\Lambda_x\Lambda|_{x=0}^{x=l} + c \int_0^l \Lambda_x^2 dx.
\] (9)

Because of the boundary conditions, \(\Lambda = 0\) at \(x = 0, l\).
\[
\frac{d}{dt} \int_0^l \left[\Lambda(x, t)\right]^2 dx = -c \int_0^l [\Lambda_x(x, t)]^2 dx \leq 0.
\] (10)

Since \(\int \Lambda^2 dx\) is non-increasing, the equation (10) becomes
\[
\int_0^l [\Lambda(x, t)]^2 dx \leq \int_0^l [\Lambda(x, 0)]^2 dx, \ t \geq 0.
\] (11)

By imposing the initial condition of \(u\), the right side of equation (11) vanishes. It implies that \(\int [\Lambda(x, t)]^2 dx = 0\) for all \(t > 0\). Consequently, \(\Lambda \equiv 0\). It follows that \(u_1 = u_2\) for all \(t \geq 0\). Therefore, the solution we obtained is unique.
On the Fourier transform

References


Received: May 1, 2014