Implementation of Richardson’s Arms Race Model
in Advertising Expenditure of Two Competitive Firms

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Abstract

The purpose of this study is the implementation of some of the most widely known mathematical theories of war in firms. During World War II mathematical models in military operations were widely used. Through this process operational research was created. In this specific research, Lewis Richardson’s arms race model was examined. This particular mathematical model was based on differential equations and its main purpose was to explain the course of the cost of equipment of two warring states. The environment in which firms operate today is highly competitive and highly fluctuating, quite similar to that of a military conflict. After the appropriate theoretical conditions were set, the above models were applied in cases of operation of modern firms. More specifically, Richardson’s arms race model was applied to secondary data from the mobile phone industry in Greece. The results of applying the model led to the conclusion that the theoretical models are almost identical with reality, which means that they can be applied to firms under the appropriate preconditions.

Keywords: Operations research, Lewis Richardson, mathematical theories of war, differential equations, advertisement in duopoly
1 Introduction

During World War II mathematical models were widely used in military operations. George Dantzig was the pioneer who released in 1947 the Simplex method, which is the first complete algorithm for solving linear programming problems (Kivotos & Frangos, 2009). Apart from linear programming, many more theories and models that were designed to solve various problems about war operations were developed. Through this process operations research was created.

This study is part of the scientific field of operations research, as it examines the applicability in firms of the mathematical models that were used during wars. Specifically, the applicability of Richardson’s arms race model will be examined in the case of the diachronic evolution of advertising expenditure in the mobile phone industry in Greece.

Lewis Richardson was born in 1881 in Newcastle and studied mathematical psychology (Lynch, 2008). He dealt with various areas of mathematical predictions, such as building models for weather forecast (Lynch, 2008). He even studied the war conflicts between states and built mathematical models in order to describe how they increase their equipment (Lynch, 2008).

Richardson believed that each state increases steadily its equipment like it is obliged to do so, something that might be due to primordial instincts, or to the lack of spiritual and moral background in order to set limits (Daras, 2001). Based on this hypothesis he built the mathematical arms race model.

In arms race model, the functions $x = x(t)$ and $y = y(t)$ represent the equipment of two states as a function of time. The rate of change depends on martial readiness and from the differences that exist between the two states. Therefore applies the system:

\[
\begin{align*}
\frac{dx}{dt} &= ky - ax + g \\
\frac{dy}{dt} &= \lambda x - by + h
\end{align*}
\]

where, $a$ and $b$ are the costs of equipment and $g$ and $h$ are the differences between states.

The applicability of the model above can be seen by comparing the equations of system (1) with the corresponding models of differential equations which were developed for the case of the effect of advertising on consumers. One such model is the simple model of advertising in Nervone and Arrow, known as N-A, which was compiled in 1962 (Little, 1979). In this model, the advertising cost of a firm is symbolized by $q(t)$, while its effect on consumers is symbolized by $A(t)$ (Little, 1979).

This model, that relates the advertising expenditure of a firm as a function of time and the corresponding effect on consumers, is given by the differential equa-
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The general solution of the differential equation that describes the model N-A, is given by
\[ A(t) = \frac{b \int e^{kt} q(t) \, dt + c}{e^{kt}}. \]

In this research the structure of Lewis Richardson’s arms race model was followed, which was adapted and then under the proper establishment of necessary theoretical conditions, its implementation was examined in secondary data that describe the advertising costs of the three largest mobile firms that operate in the Greek market during the period 2000-2006.

2 Construction of the mathematical model

We consider two competitive firms A and B which coexist in a common market and sell the product P at the same price. This market is characterized as an oligopoly, as we deem that there aren’t other firms that sell the same product and the input of such firms is very difficult. At the same time, the market is closed and it is certain that international trade is not conducted.

Still, there are no substitutes for the product P, while the introduction of firms into the market that sell substitute products is excluded. Finally, it is accepted that if one of the two firms does not spend resources on advertising, then it is unprofitable for the other to continue spending such resources. This is confirmed, as according to Bester & Petrakis (1995) advertising attaches in an oligopoly only in case where sellers offer their product at the same price, while in the case of different prices, consumers choose the product that costs less.

Assuming the function that represents the advertising cost of a firm A and supposing the function that represents the corresponding expense competitive to firm A, firm B.

The rate of change of the function \( x(t) \) and \( y(t) \) depends on the competition between the two firms in various markets in which they coexist. Thus, the rate of change of \( x(t) \) for firm A depends on how ready to compete is firm B, as well as the demands of the market share. We denote these terms as \( kx \) and \( g \) respectively, where \( k \) and \( g \) are suitable constants. These terms cause the increase of the price of \( x \).

Correspondingly, the rate of change of \( y(t) \) for firm B depends on how ready to compete is firm A, as well as its claims. We denote these terms as \( \lambda x \) and \( h \) respectively, where \( \lambda \) and \( h \) are suitable constants. These terms cause the increase of the price of \( y \).

The total size of the advertising expenditure of the two competitive firms constitutes the term for firm A and the term for firm B. These sizes are expressed with \( -ax \) and \( -by \) respectively, where \( a \) and \( b \) are suitable positive constants.
The two time functions constitute the solution of the following system of differential equations:

\[
\begin{align*}
\frac{dx}{dt} &= ky - ax + g \\
\frac{dy}{dt} &= \lambda x - by + h
\end{align*}
\]  

(2)

Assuming that \( g = 0 \) and \( h = 0 \) then, functions \( x(t) = 0 \) and \( y(t) = 0 \) form a solution for the system of differential equations (2). In this case, function (1) is balanced meaning that if \( x, y, g \) and \( h \) all equal to zero, then the functions \( x(t) \) and \( y(t) \) will remain zero for the entire duration.

In the case of firms A and B, the above situation means that none of these two firms invest money in advertising in the competition between them.

Thus, assuming that \( x \) and \( y \) are zeroed at the time \( t_0 = 0 \) then at this time it will apply that \( \frac{dx}{dt} = g \) and \( \frac{dy}{dt} = h \) this means that \( x \) and \( y \) will not continue to be equal to zero, since \( g \) and \( h \) are now positive numbers.

If one of the competitive firms A and B decide not to invest money funds in advertising, \( x \) or \( y \) respectively will be zeroed at a certain time \( t_i \). Thus, for firm A at some time \( t_i \) it will apply that \( \frac{dx}{dt} = \lambda x + h \) and respectively for firm B it will apply that \( \frac{dy}{dt} = k x + g \).

In the case where \( a, b, g \) and \( h \) are all equal to zero, system (2) will be written in the form:

\[
\begin{align*}
\frac{dx}{dt} &= ky \\
\frac{dy}{dt} &= \lambda x
\end{align*}
\]  

(3)

Any solution of the system of differential equations (3) will be given by the formulas:

\[
x(t) = \frac{1}{2} c_1 e^{-\sqrt{k}\sqrt{\lambda} t} \left( e^{2\sqrt{k}\sqrt{\lambda} t} + 1 \right) + \frac{c_2 \sqrt{k} e^{-\sqrt{k} \sqrt{\lambda} t} \left( e^{2\sqrt{k}\sqrt{\lambda} t} - 1 \right)}{2\sqrt{\lambda}} 
\]  

(4)

\[
y(t) = \frac{c_1 \sqrt{\lambda} e^{-\sqrt{k}\sqrt{\lambda} t} \left( e^{2\sqrt{k}\sqrt{\lambda} t} - 1 \right)}{2\sqrt{k}} + \frac{1}{2} c_2 e^{-\sqrt{k}\sqrt{\lambda} t} \left( e^{2\sqrt{k}\sqrt{\lambda} t} + 1 \right)
\]  

(5)

The solution of the system of differential equations (3) extract in the formulas (4) and (5) the coefficients \( c_1 \) and \( c_2 \). These coefficients are equal to \( x(t) \) and \( y(t) \) respectively for \( t = 0 \). This means that at the time \( t_0 = 0 \), it applies that \( c_1 = x(t) \) and \( c_2 = y(t) \).

The above situation, where \( x(t) \) and \( y(t) \) tend to infinity, can be interpreted as a situation where firms are constantly competing with each other.
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Figure 1, shows exactly this situation, namely the constant competition between the two firms and the advertising cost which for both A and B is ascending and tends to infinity. The chart also shows that firm B spends more money than A for advertising.

\[ \text{Figure 1. The graphical representation of the solution of the system of differential equations (2), where } c_1 = 1, c_2 = 2, k = 1 \text{ and } \lambda = 2. \]

At this point of the study, it is interesting to examine for which \( c_1, c_2, k \) and \( \lambda \) the curves of Figure 1 intersect, namely for which \( c_1, c_2, k \) and \( \lambda \) the advertising expenditures of firm A surpass the B ones at some time \( t \). It is also interesting the measurement of dispute arising between the expenditures of the two firms for advertising.

The image of the curves of Figure 1 and the symmetry of the two equations give the answer to what \( c_1, c_2, k \) and \( \lambda \) the curves of Figure 1 intersect. Keeping \( k \) and \( \lambda \) in combination with the reversal of \( c_1 \) and \( c_2 \) for the two equations, leads eventually to the answer of the question above, which is also confirmed by Figure 2 where the curves \( x(t) \) and \( y(t) \) intersect at the time \( t = 0.85 \).

\[ \text{Figure 2. The graphical representation of the solution of the system of differential equations (3) where } c_1 = 1, c_2 = 2, k = 1 \text{ and } \lambda = 2. \]

In order to figure out the difference between the advertising expenditure of the two firms, the area enclosed between the functions \( x(t) \) and \( y(t) \) will be calculated.

The calculation will be done with the help of certain integrals for the two functions for \( t = 0 \) up to \( t = 0.85 \) which is the time that corresponds to the moment where the two functions intersect i.e. by using
The difference between the two areas is equal to 0.89 which corresponds to the difference between the advertising expenditure of firm A and the advertising expenditure of firm B for \( t = 0 \) up to \( t = 0.85 \).

Therefore, based on the above calculation, firm B spends a total of 0.85 further monetary units for advertising in comparison with firm A, up to the time \( t = 0.85 \).

At this point of the analysis of Richardson’s arms race model, Daras (2001) notes that the corresponding system of differential equations (3) can not be considered complete, since it is not included in this international trade which may be carried, or any other international cooperation with benefits for both countries to engage in warfare. Thus, it is proposed that in the under consideration system of differential equations, in order for it to be complete, the profit must be removed from the cooperation between the two countries, if they go to war.

The same thing happens in the case of firms. It is possible, for two competitive firms to cooperate with each other in order to have common benefits (Papadakis, 2012).

This is something that is also confirmed from the collaboration that existed until 2011 between Apple and Samsung in smartphone construction, which was terminated after a legal battle due to the competition between them (Chou, 2013; Bosker & Grandoni, 2012).

Therefore, it is possible to do the same analysis in the case of competition between firms A and B.

We set as \( U \) the budget of firm A for its advertising campaign and \( U_0 \) the economic benefits of the transaction with firm B, before the end of the collaboration. Accordingly, we set \( V \) the budget of firm B for its advertising campaign and \( V_0 \) the economic benefits of the transaction with firm A before the end of the collaboration.

Based on the new data, we deem the dependent variables \( x = U - U_0 \) and \( y = V - V_0 \). In this case, the following functions constitute the solution of the system (2):

\[
x = x_0 = \frac{kh + bg}{ab - k\lambda} \quad (6)
\]

\[
y = y_0 = \frac{\lambda g + ah}{ab - k\lambda} \quad (7)
\]

where, \( ab - k\lambda \neq 0 \).

In order to prove mathematically that firms A and B will compete against each other, as in the case of countries that will go into warfare, it should be invest-
igated whether the fact of the equilibrium solution of system (6) is stable or not. To enable the above investigation, system (2) will reformulate as follows:

\[
\begin{align*}
\dot{W} &= AW + F \\
W &= W(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \\
\dot{W} &= \dot{W}(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}, \\
A &= \begin{pmatrix} -a & k \\ \lambda & -b \end{pmatrix} \text{ and, } \\
F &= \begin{pmatrix} g \\ h \end{pmatrix}.
\end{align*}
\]

The equilibrium system solution will be \( W = W_o = \begin{pmatrix} x_o \\ y_o \end{pmatrix} \) and verifies the equation \( AW_o + F = 0 \). Thereafter setting \( Z = Z(t) = W - W_o \), we have \( \dot{Z} = \dot{W} = AW + F = A(Z + W_o) = AZ + AW_o + F = AZ \).

Therefore follows that the equilibrium solution \( \dot{W} = AW + F \) of equation (7) is stable in the case where the vector function \( Z = 0 \) is a stable solution of the equation \( Z = AZ \).

In order to determine the stability of \( Z = 0 \) the following polynomial must be calculated \( p(\lambda) = \det(\begin{pmatrix} -a - \lambda & k \\ \lambda & -b - \lambda \end{pmatrix}) = \lambda^2 + (a + b)\lambda + (ab - k\lambda). \) The square roots of \( p(\lambda) \) are \( \lambda = \frac{-(a + b) \pm [(a + b)^2 + 4k\lambda]^\frac{1}{2}}{2} \).

Therefore it is perceived that both square roots of \( p(\lambda) \) are real and different from zero, negative when \( ab - k\lambda > 0 \), while one of the two is negative when \( ab - k\lambda < 0 \). This entails that \( Z(t) = 0 \) and thus the balanced solution is stable in the case where \( ab - k\lambda > 0 \), while it is unstable in the case where \( ab - k\lambda < 0 \).

Richardson considered as \( a \) and \( b \) the average lifetime of governments of countries that are at war (Daras, 2001). Accordingly, in the case of firms, the parameters \( a \) and \( b \) can be considered as the average lifespan of their administrations.

In order to calculate the values of \( k \) and \( \lambda \), we consider \( g = 0 \) and \( y = y_1 \) so that \( \frac{dx}{dt} = ky_1 - ax \). When \( x = 0 \), it will apply that \( \frac{1}{k} = \frac{y_1}{\frac{dx}{dt}} \) so that \( \frac{1}{k} \) will be eventually the minimum time that firm A requires in order to spend the same amount of money with firm B.

### 3 Implementation of the mathematical model

To test the mathematical model that derived from the analysis above, it will be applied using real data under the theoretical conditions that were set at the beginning of its construction.

The implementation of the mathematical model will be held in the Greek mobile telephone market.

This market is comprised of three firms, Cosmote which owns the largest share over time (Chatzivassiliadou, 2008), Vodafone and Wind.
is one of the most important technological sectors in Greek economy and contributes to national income growth, the increase of government revenue and the creation of new jobs (Drosos et al., 2011).

This market develops very quickly and is led to its maturation process (Santouridis & Trivellas, 2010). Firms that are active in mobile telephony in Greece offer their services at similar prices, this means that ad campaigns play a key role in shaping demand since they affect significantly the costumers (Kleiousis, 2012). The competition between firms in the market, is particularly fierce and in conjunction with its characteristic features, it takes the form of oligopoly which may lead to partnerships and even mergers (Michalakelis et al., 2008). All the above indicate that the structure of this market closely approximates the theoretical formulation of the mathematical model and therefore is suitable for its application.

To make it possible to analyze the data based on the conditions of the mathematical model, which involves two competitive firms or two competitive coalition firms, in line with Richardson’s original theory of involving two warring countries or two opposing country alliances, it will be considered that two of the three firms ally with each other. Besides this scenario is as mentioned likely to occur due to intense competition and the structure of this market, we assume therefore that there is an alliance between Vodafone and Wind.

The first issue in the implementation of the model is to determine whether there is real competition between firms.

Based on the data of market shares, Cosmote will be roughly equivalent to a potential partnership between Vodafone and Wind so we deem that \( k = \lambda = 0.9 \).

We also consider that the average lifespan of firms management are identical and equal to three years. So we have \( a = b = 0.33 \).

Given the above, system (2) of differential equations has a unique equilibrium point \( W_0 = \left( x_0, y_0 \right) \), where 
\[
x_0 = \frac{kh + ag}{a^2 - k^2} \quad \text{and} \quad y_0 = \frac{kg + ah}{a^2 - k^2}.
\]

The above balance is proved to be unstable since 
\[
ab - k\lambda = a^2 - k^2 = 0.11 - 0.81 = -0.7, \quad \text{which means that} \quad ab - k\lambda < 0.
\]
This result is therefore compatible with the fact that in this case there is competition between Cosmote and the alliance of Vodafone with Wind.

Table 1 lists the advertising budgets of firms during the period between 2000 and 2006.
Table 1. Advertising expenditure (in millions of €) of mobile telephony firms in Greece during the period 2000-2006. Source: Chatzivassiliadou, 2008.

<table>
<thead>
<tr>
<th></th>
<th>2000</th>
<th>2001</th>
<th>2002</th>
<th>2003</th>
<th>2004</th>
<th>2005</th>
<th>2006</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wind</td>
<td>16.193</td>
<td>17.079</td>
<td>18.879</td>
<td>17.456</td>
<td>23.755</td>
<td>32.011</td>
<td>34.670</td>
</tr>
<tr>
<td>Total</td>
<td>44.704</td>
<td>54.050</td>
<td>66.692</td>
<td>69.618</td>
<td>92.433</td>
<td>100.212</td>
<td>102.773</td>
</tr>
</tbody>
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<tbody>
<tr>
<td>Δ(U+V)</td>
<td>9.345</td>
<td>2.926</td>
<td>7.785</td>
</tr>
<tr>
<td>U+V in the same year</td>
<td>49.377</td>
<td>68.155</td>
<td>96.326</td>
</tr>
</tbody>
</table>

From the data of Table 1, Figure 3 indicates which describes the annual increment of the sum U + V for the two years that each increment needs.

Figure 3. d(U+V) as a function of U+V for each year.

The points corresponding to the sum U+V for each year, namely the money spent every year are very close to the line $d(U+V) = 7.896 + 0.17(U+V)$. Correlation coefficient $R^2 = 0.924$ certifies the high correlation of data and the suitability of the predictions of the model.

4 Results

From the application of the model to the above data, it was concluded that a theory like Lewis Richardson’s arms race model can be applied in a situation that concerns modern firms.

The good adaptation of the mathematical arms race model, proves that modern markets where firms compete with one another can be seen as a warfare on a battlefield, only if one of the firms manages to convince consumers to buy their own products and become victorious. (Taoka, 1997). In this case we don’t count the battles won, but their market share (Taoka, 1997).
Therefore it is possible models of mathematical theories of war to be applied after suitable modifications and the adoption of appropriate theoretical conditions in cases relating to modern firms. In conclusion it is proposed to further explore the implementation of such models as the battle models of Lanchester in cases like those considered in this study.

References


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