Vertex Graceful Labeling of $C_j \cup C_k \cup C_l$

P. Selvaraju $^1$, P. Balaganesan $^{2,5}$, J. Renuka $^3$ and M.L. Suresh $^4$

$^1$Department of Mathematics  
Vel Tech Multi Tech  
Dr. Rangarajan Dr. Sankanthula  
Engineering College  
Avadi, Chennai- 600 062, India  

$^2$Research Scholar  

$^2, 4$Department of Mathematics  
Hindustan University  
Chennai- 603 103, India  

$^3$Departments of Mathematics  
Sri Sai Ram Engineering College  
Chennai - 600 044, India  

$^5$Department of Mathematics  
Saveetha School of Engineering  
Saveetha University  
Chennai- 602 105, Tamil Nadu, India

Copyright © 2014 P. Selvaraju, P. Balaganesan J. Renuka, M.L. Suresh. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

The disjoint union $C_j \cup C_k \cup C_l$ is component of three cycles $C_j, C_k$ and $C_l$. In this paper, we prove that the disjoint union graph $C_j \cup C_k \cup C_l$ is vertex graceful for odd $j + k + l$ with $l \geq j + k + 5$.

Mathematics Subject Classification: 05C78

Keywords: Vertex graceful, disjoint of Union cycles, vertex labeling, edge labeling
1 Introduction

Graph labeling, where the vertices are assigned values subject to certain conditions have often been motivated by practical problems. Labeled graphs serves as useful mathematical models for a broad range of applications such as coding theory, including the design of good radar type codes, synch-set codes, missile guidance codes and convolution codes with optimal auto correlation properties. They facilitate the optimal nonstandard encoding of integers.

All graphs in this paper are finite, simple graphs and undirected graph. The symbols $V(G)$ and $E(G)$ denote the vertex set and edge set of a graph $G$. The cardinality of the vertex set is called the order of $G$. The cardinality of the edge set is called the size of $G$. A graph with $p$ vertices and $q$ edges is called a $G(p, q)$ graph.

Most graph labeling methods trace their origin to one introduced by Rosa [1] called such a labeling a $\beta$-valuation and Golomb [2] subsequently called graceful labeling, and one introduced by Graham and Sloane [3] called harmonious labeling. Several infinite families of graceful and harmonious graphs have been reported. Many illustrious works on graceful graphs brought a tide to different ways of labeling the elements of graph such as odd graceful.

A graph $G$ with $p$ vertices and $q$ edges is said to be vertex graceful if there exists labeling $f : V(G) \to \{1, 2, 3, \cdots, p\}$ such that the induced labeling $f^+ : E(G) \to \mathbb{Z}$ defined by $f^+(u, v) = f(u) + f(v) \mod q$ is a bisection. The concept of vertex graceful was introduced by Lee, Pan and Tsai in 2005 [4]. In this paper, we show that the disjoint union of three cycles with odd order, namely $C_j \cup C_k \cup C_l$ is vertex graceful for odd $j + k + l$ with $l \geq j + k + 5$.

2 Main Results

Theorem 2.1. The disjoint union $C_j \cup C_k \cup C_l$ of cycles $C_j, C_k$ and $C_l$ is vertex graceful for odd for odd $j + k + l$ with $l \geq j + k + 5$.

Proof. Let $G$ be the disjoint union on $n$ vertices and $q$ edges, then $n = q = j + k + l$. Let $V(G) = \{u_i : 0 \leq i \leq j - 1\} \cup \{v_i : 0 \leq i \leq k - 1\} \cup \{w_i : 0 \leq i \leq l - 1\}$, $E(G) = \{u_iu_{(i+1) \mod j} : 0 \leq i \leq j - 1\} \cup \{v_iv_{(i+1) \mod k} : 0 \leq i \leq k - 1\} \cup \{w_iw_{(i+1) \mod l} : 0 \leq i \leq l - 1\}$.

$\theta = \left(\frac{l-j-k-3}{2}\right) \mod 2$. By symmetry, we only need to consider three cases as below:

Case (i): $k, l$ are even and $j$ is odd.

We label the vertices as follows:

$$f(u_i) = \begin{cases} 2i + 1, & 0 \leq i \leq \left(\frac{j-1}{2}\right) \\ 2(j - i), & \left(\frac{j+1}{2}\right) \leq i \leq j - 1 \end{cases}$$
\[ f(v_i) = \begin{cases} 
q - 2i, & 0 \leq i \leq \left(\frac{k}{2}\right) - 1 \\
q - 2(k - i) + 1, & \left(\frac{k}{2}\right) \leq i \leq k - 1
\end{cases} \]

\[ f(w_i) = \begin{cases} 
j + 1, & i = 0 \\
j + 2i + \theta, & 1 \leq i \leq \frac{l - j - k - 3 - 2\theta}{4} \\
j + 2i + 4 + \theta, & \frac{l - j - k - 1 - 2\theta}{4} \leq i \leq \frac{l - j + k - 7 + 2\theta}{4} \\
j + 2i + 4 + \theta, & \frac{l - j + k - 3 - 2\theta}{4} \leq i \leq \frac{l + j + k - 9 - 2\theta}{4} \\
j + 2i + 3 + \theta, & i = \frac{l + j + k - 5 - 2\theta}{4} \\
j + 2i + 4 + \theta, & \frac{l + j + k - 1 - 2\theta}{4} \leq i \leq \frac{l + j + k - 7 + 2\theta}{4} \\
j - 1 + 2(l - i) - \theta, & \frac{l + j + k - 3 - 2\theta}{4} \leq i \leq \frac{l + j + k - 9 - 2\theta}{4} \\
j - 2 + 2(l - i) - \theta, & i = \frac{l + j + k - 5 - 2\theta}{4} \\
j - 1 + 2(l - i) - \theta, & \frac{l + j + k - 1 - 2\theta}{4} \leq i \leq \frac{l + j + k - 3 - 2\theta}{4} \\
j + 2(l - i) - \theta, & i = \frac{l + j + k - 5 - 2\theta}{4} \\
j + 1 + 2(l - i) - \theta, & \frac{l + j + k - 3 - 2\theta}{4} \leq i \leq l - 1
\end{cases} \]

Let \( D \) be the label set of all edges, then we have \( D = D_1 \cup D_2 \cup D_3 \), where \( D_1 = \{u_i u_{(i+1) \mod j} : 0 \leq i \leq j - 1\} \), \( D_2 = \{v_i v_{(i+1) \mod k} : 0 \leq i \leq k - 1\} \), \( D_3 = \{w_i w_{(i+1) \mod l} : 0 \leq i \leq l - 1\} \). It is obvious that the labels of each edge are different. So \( g \) maps \( E \) onto \( \{0, 1, 2, \ldots, (q - 1)\} \). According to the definition of vertex graceful labeling, we can conclude that the disjoint union cycles \( C_j \cup C_k \cup C_l \) is vertex graceful for \( k, l \) are even and \( j \) is odd and \( l \geq j + k + 5 \).

![Figure 1](image-url)

**Case (ii):** \( j, k \) are even and \( l \) is odd.

We label the vertices as follows:
\[ f(u_i) = \begin{cases} 
2i + 1, & 0 \leq i \leq \left(\frac{j}{2}\right) - 1 \\
2(j - i), & \left(\frac{j}{2}\right) \leq i \leq j - 1 
\end{cases} \]

\[ f(v_i) = \begin{cases} 
q - 2i, & 0 \leq i \leq \left(\frac{k}{2}\right) - 1 \\
q - 2(k - i) + 1, & \left(\frac{k}{2}\right) \leq i \leq k - 1 
\end{cases} \]

\[ f(w_i) = \begin{cases} 
j + 1, & i = 0 \\
j + 2i + \theta, & 1 \leq i \leq \frac{l - j - k - 3 - 2\theta}{4} \\
j + 2i + 2 + \theta, & \frac{l - j - k - 3 - 2\theta}{4} \leq i \leq \frac{l - j - k - 7 - 2\theta}{4} \\
j + 2i + 4 + \theta, & \frac{l - j - k - 7 - 2\theta}{4} \leq i \leq \frac{l + j + k - 11 - 2\theta}{4} \\
j + 2i + 6 + \theta, & \frac{l + j + k - 11 - 2\theta}{4} \leq i \leq \frac{l + 7}{2} 
\end{cases} \]

By an argument similar to the one in case 1, we have this assignment provides a vertex graceful labeling for \( j, k \) are even and \( l \) is odd and \( l \geq j + k + 5 \).

**Case (iii):** \( j, k, l \) are all odd.

We label the vertices as follows:
Vertex graceful labeling of $C_j \cup C_k \cup C_l$

$f(u_i) =$ \begin{cases} 
2i + 1, & 0 \leq i \leq \frac{j - 1}{2} \\
2(j - i), & \frac{j + 1}{2} \leq i \leq j - 1 
\end{cases}

$f(v_i) =$ \begin{cases} 
q - 2i, & 0 \leq i \leq \frac{k - 1}{2} \\
q - 2(k - i) + 1, & \frac{k + 1}{2} \leq i \leq k - 1 
\end{cases}

\begin{align*}
&f(w_i) = \begin{cases} 
j + 1, & i = 0 \\
\frac{j + 2i + \theta}{4}, & 1 \leq i \leq \frac{l - j - k - 3 - 2\theta}{4} \\
\frac{j + 2i + 2 + \theta}{4}, & \frac{l - j - k + 2\theta}{4} \leq i \leq \frac{l - j + k - 5 - 2\theta}{4} \\
\frac{j + 2i + 1 + \theta}{4}, & i = \frac{l - j + k - 1 - 2\theta}{4} \\
\frac{l - j + \theta + \frac{3l + j - k - 3 - 2\theta}{4}}{4}, & \frac{l - j + \theta}{4} \leq i \leq \frac{l + j + k - 2 - 2\theta}{4} \\
\frac{j + 2i + 2}{4}, & i = \frac{l - j + k - 2 - 2\theta}{4} \\
\frac{l + 2(l - i) - \theta}{4}, & 3\frac{l - j - k - 2\theta}{4} \leq i \leq \frac{l - j - k + 3 - 2\theta}{4} \\
\frac{j + 1 + 2(l - i) - \theta}{4}, & \frac{3l + j + k - 1 - 2\theta}{4} \leq i \leq \frac{3l + j - k + 3 - 2\theta}{4} \\
\frac{j + 1 + 2(l - i) - \theta}{4}, & i = \frac{3l + j + k - 2 - 2\theta}{4} \\
\frac{j + 1 + 2}{4}, & 3\frac{l + j + k + 3 - 2\theta}{4} \leq i \leq l - 1 \\
\frac{j + 1 + 2(l - i) - \theta}{4}, & i = \frac{3l + j + k - 1 - 2\theta}{4} \\
\frac{j + 1 + 2(l - i) - \theta}{4}, & 3\frac{l + j + k - 3 - 2\theta}{4} \leq i \leq l - 1 
\end{cases}
\end{align*}

Since the proof in this case is similar to the one in case 1, we omit it. Thus, we have that this assignment provides a vertex graceful labeling for $C_j \cup C_k \cup C_l$, when $j, k, l$ are all odd and $l \geq j + k + 5$. Therefore, we can conclude that the graph of the disjoint union of $C_j \cup C_k \cup C_l$ is vertex graceful for odd $j + k + l$ with $l \geq j + k + 5$. \qed

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{vertex_graceful_labeling}
\caption{Figure 3:}
\end{figure}
**Conjecture:** The disjoint union of $C_j \cup C_k \cup C_l$ is vertex graceful for any $j, k, l$.

**References**


**Received:** May 9, 2014