The \( \text{coth}_a(\xi) \) Expansion Method and its Application to the Davey-Stewartson Equation

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Abstract

In this article we propose a new method called \( \text{coth}_a(\xi) \) Expansion method that uses the generalized hyperbolic functions, to find the exact solutions of Davey-Stewartson nonlinear partial differential equation. By taking different parameter values of the generalized hyperbolic functions, we found some special solutions. The solutions generated by the proposed method are compared with the ones obtained by some existing methods to show its efficiency. The proposed method can be used to solve other nonlinear partial differential equations.

Keywords: Generalized Hyperbolic Function, Davey-Stewartson Equation, Exact solutions
1 Introduction

In the nonlinear science, many important phenomena in various fields can be described by the nonlinear evolution equations (NLEEs). The study of exact solutions, especially traveling wave solutions, for NLPDEs plays a significant role in the study of nonlinear physical phenomena. These exact solutions when they exist can help to understand the dynamical processes that are modeled by the corresponding nonlinear evolution equations (NLEEs). In this paper, we will study the Davey-Stewartson equation (DSE) which arises in the study of fluid dynamics.

In fact, this equation particularly studies the long-wave-short-wave resonances and other patterns of propagating waves [1,2]. Also, this equation describes the evolution of a 3-dimensional wave-packet on water of finite depth. Some solutions for this equation can be found in [3,4].

In recent years, many powerful methods to construct exact solutions of nonlinear evolution equations have been established and developed such as the inverse scattering transform method [5], the Hirota method [6], the Backlund transform method [7], the exp-function method [8], truncated Painleve expansion method [9], the Weierstrass elliptic function method [10], the tanh-function method [11] and the Jacobi elliptic function expansion method [12,13]. Also some authors define new functions which are called generalized hyperbolic functions for constructing new solutions [14,15].

2 The Definition and Properties of the Symmetrical Hyperbolic Fibonacci and Lucas Function

In this section, we will define new functions which are named the symmetrical hyperbolic Fibonacci and Lucas functions for constructing new exact solutions of NPDEs and then study the properties of these functions.

Definition (2-1)
Suppose that ξ is an independent variable, p, q and k are all constant. The generalized hyperbolic sine function is

\[ \sinh_a(\xi) = \frac{(pa^k - qa^{-k})}{2} \] (1)

generalized hyperbolic cosine function is

\[ \cosh_a(\xi) = \frac{(pa^k + qa^{-k})}{2} \] (2)
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generalized hyperbolic tangent function is

$$\tanh_a(\xi) = \left( \frac{pa^k\xi - qa^{-k\xi}}{pa^k\xi + qa^{-k\xi}} \right)$$

(3)

generalized hyperbolic cotangent function is

$$\coth_a(\xi) = \left( \frac{pa^k\xi + qa^{-k\xi}}{pa^k\xi - qa^{-k\xi}} \right)$$

(4)

**Definition (2-2)**

The derivative formulae of generalized hyperbolic functions as following:

$$\frac{d \left( \sinh_a(\xi) \right)}{d\xi} = k\ln(a) \cosh_a(\xi)$$

(5)

$$\frac{d \left( \cosh_a(\xi) \right)}{d\xi} = k\ln(a) \sinh_a(\xi)$$

(6)

$$\frac{d \left( \tanh_a(\xi) \right)}{d\xi} = k\ln(a) \left( 1 - \tanh_a^2(\xi) \right)$$

(7)

$$\frac{d \left( \coth_a(\xi) \right)}{d\xi} = k\ln(a) \left( 1 - \coth_a^2(\xi) \right)$$

(8)

**Remark 1** We see that when \( p = 1, q = 1, k = 1 \) and \( a = e \) in (1)-(4), new generalized hyperbolic function \( \sinh_a(\xi), \cosh_a(\xi), \tanh_a(\xi) \) and \( \coth_a(\xi) \) degenerate as hyperbolic function \( \sinh(\xi), \cosh(\xi), \tanh(\xi) \) and \( \coth(\xi) \) respectively. In addition, when \( p = 0 \) or \( q = 0 \) in (1)-(4), \( \sinh_a(\xi), \cosh_a(\xi), \tanh_a(\xi) \) and \( \coth_a(\xi) \) degenerate as exponential function \( \frac{1}{2}pa^{k\xi}, \pm qa^{-k\xi}, \) and \( \pm 1 \) respectively.

3 Description Of the \( \text{coth}_a(\xi) \) Expansion method

Suppose that a nonlinear evolution equation is given by

$$F(u, u_t, u_x, u_y, u_{xx}, u_{tt}, u_{yy}, u_{xt}, u_{xy} \ldots \ldots) = 0,$$

(9)
where \( u = u(x, y, t) \) is an unknown function, \( F \) is a polynomial in \( u \) and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following, we give the main steps of this method:

**Step 1:** Using the generalized wave transformation

\[
\begin{align*}
  u(x, y, t) &= u(\xi), \quad \xi = \alpha x + \beta y + \gamma t, \\
  \text{where} \quad \alpha, \beta \quad \text{and} \quad \gamma \quad \text{are a constant,}
\end{align*}
\]

Then, Eq. (9) is reduced to the following ODE:

\[
P\left(u, \alpha u', \beta u', \gamma u', \alpha^2 u'', \beta^2 u'', \gamma^2 u'' \ldots \right) = 0,
\]

(11)

where \( \left( = \frac{d}{d\xi} \right) \) and \( P \) is a polynomial in \( u \) and its total derivatives.

**Step 2:** We suppose that Eq. (11) has the following formal solution:

\[
  u(\xi) = \sum_{i=0}^{N} \alpha_i \left( \frac{A_1 \coth_a(\xi) + A_2}{B_1 \coth_a(\xi) + B_2} \right)^i
\]

(12)

where \( N \) is a positive integer, and \( A_1, A_2, B_1, B_2, \alpha_i (i = 1, 2, \ldots) \) are constants, and the function \( \coth_a(\xi) \) satisfies the following derivatives:

\[
\begin{align*}
  \frac{d}{d\xi} (\coth_a(\xi)) &= k \ln(a) (1 - \coth_a^2(\xi)) \\
  \frac{d^2}{d\xi^2} (\coth_a(\xi)) &= -2k^2 \ln^2(a) \coth_a(\xi) (1 - \coth_a^2(\xi)) \\
  \frac{d^3}{d\xi^3} (\coth_a(\xi)) &= 2k^3 \ln^3(a) (3 \coth_a^2(\xi) - 1) (1 - \coth_a^2(\xi)) \\
  \frac{d^4}{d\xi^4} (\coth_a(\xi)) &= -8k^4 \ln^4(a) \coth_a(\xi) (3 \coth_a^2(\xi) - 2) (1 - \coth_a^2(\xi))
\end{align*}
\]

(13)

and so on.

**Step 3:** Determine the positive integer \( N \) in (12) by balancing the highest order derivatives and nonlinear terms in Eq. (11).

**Step 4:** Substituting (12) along with (13) into (11) and then setting all the coefficients of \( (\coth_a(\xi))^i \) \((i = 0, 1, 2, \ldots)\) of the resulting system’s numerator to zero, yields a set of over
determined nonlinear algebraic equations for $\alpha, \beta, \gamma, A_1, A_2, B_1, B_2$ and $\alpha_i (i = 0, 1, 2, \ldots N)$.

**Step 5:** Solving these algebraic equations by Maple or Mathematica, we get the values of $\alpha, \beta, \gamma, A_1, A_2, B_1, B_2$ and $\alpha_i (i = 0, 1, 2, \ldots N)$.

**Step 6:** Substituting these values into (12), we can obtain the exact traveling wave solutions of Eq. (11).

### 4 The New Exact Solutions of Davey-Stewartson Equation

In this section, we use the ($\coth_a (\xi)$) expansion method to construct the exact solutions of the Davey-Stewartson equation, we consider the Davey-Stewartson equations in the following form:

\[
\begin{align*}
(iW_t + \delta_0 W_{xx} + W_{yy}) &= \delta_1 |W|^2 W + \delta_2 WF_x \\
F_{xx} + \delta_3 F_{yy} &= (|W|^2)_x
\end{align*}
\]  

(14)

We may choose the following traveling wave transformation:

\[
\begin{align*}
W &= u(\xi) e^{i\varphi}, F(\xi) = v(\xi) \\
\xi &= k_1 x + k_2 y - 2(k_2 l_2 + \delta_0 k_1 l_1) t \\
\varphi &= l_1 x + l_2 y + l_3 t
\end{align*}
\]  

(15)

where the $k_1, k_2, l_1, l_2$ and $l_3$ are arbitrary constants. Equations above become

\[
\begin{align*}
(l_3 + \delta_0 l_1^2 + l_2^2) u + \delta_1 u^3 - (\delta_0 k_1^2 + k_2^2) u'' + k_1 \delta_2 u v' &= 0 \\
(k_1^2 + \delta_3 k_2^2) v'' - 2k_1 uu' &= 0
\end{align*}
\]  

(16) \hspace{1cm} (17)
By integrating (17) with respect to \( \xi \) and setting the constant of integration to zero, we find
\[
v' = \frac{k_1}{(k_1^2 + \delta_3 k_2^2)} u^2
\] (18)

Substituting (18) into (16) yields
\[
\left( \delta_1 + \frac{\delta_2 k_1^2}{k_1^2 + \delta_3 k_2^2} \right) u^3 + \left( l_3 + \delta_0 l_1^2 + l_2^2 \right) u - \left( \delta_0 k_1^2 + k_2^2 \right) u'' = 0
\] (19)

Balancing \( u^3 \) with \( u'' \) gives \( N = 1 \). Therefore, we can write the solution of (19) in the form:
\[
u(\xi) = \alpha_0 + \alpha_1 \left( \frac{A_1 \coth_a(\xi) + A_2}{B_1 \coth_a(\xi) + B_2} \right),
\] (20)
\[
\alpha_1 \neq 0
\] (21)

where \( A_1, A_2, B_1, B_2, \alpha_1 \) and \( \alpha_2 \) are constant to be determined later. Substituting (20) along with (13) into Eq. (19) and then setting all the coefficient of \( (\coth_a(\xi))^k \) \( (k = 0, 1, 2, \ldots) \) of the resulting system’s numerator to zero, yields a set of over-determined nonlinear algebraic equations about \( A_1, A_2, B_1, B_2, \alpha_1, \alpha_2, k_1, k_2, l_1, l_2 \) and \( l_3 \). Solving the over-determined algebraic equations by Maple, we can obtain the following results:

**Case 1:**

\[
A_1 = A_2 = A_2, k_1 = k_1, k_2 = k_2, l_1 = l_1, l_2 = l_2, B_1 = 0, B_2 = B_2
\] (22)
\[
\alpha_0 = -\sqrt{\frac{2(k_1^2 + \delta_3 k_2^2) (\delta_0 k_1^2 + k_2^2)}{(\delta_1 + \delta_2) k_1^2 + \delta_1 \delta_3 k_2^2}} \left( \frac{A_2}{A_1} k \ln(a) \right)
\]
\[
\alpha_1 = \sqrt{\frac{2(k_1^2 + \delta_3 k_2^2) (\delta_0 k_1^2 + k_2^2)}{(\delta_1 + \delta_2) k_1^2 + \delta_1 \delta_3 k_2^2}} \left( \frac{B_2}{A_1} k \ln(a) \right)
\]
\[
l_3 = -2k^2 \ln^2(a) (\delta_0 k_1^2 + k_2^2) - \delta_0 l_1^2 - l_2^2
\]

Substituting (21) into (20), we have
\[
u(\xi) = \sqrt{\frac{2(k_1^2 + \delta_3 k_2^2) (\delta_0 k_1^2 + k_2^2)}{(\delta_1 + \delta_2) k_1^2 + \delta_1 \delta_3 k_2^2}} k \ln(a) \coth_a(\xi)
\] (23)
In view of (15), (18) and (22), we obtain the general solution of Eq. (4.1) in the form

\[
W = \sqrt{2 \frac{(k^2_1 + \delta_3 k^2_2) (\delta_0 k^2_1 + k^2_2)}{((\delta_1 + \delta_2) k^2_1 + \delta_1 \delta_3 k^2_2)}} k \ln (a) \coth_a (\xi)
\times \exp \left\{ i \left[ l_1 x + l_2 y - (2 k^2 \ln^2 (a) (\delta_0 k^2_1 + k^2_2) + \delta_0 l_1^2 + l_2^2) t \right] \right\}
\]

(24)

and

\[
F (\xi) = v (\xi) = \int \frac{k_1}{(k^2_1 + \delta_3 k^2_2)} \left( \sqrt{2 \frac{(k^2_1 + \delta_3 k^2_2) (\delta_0 k^2_1 + k^2_2)}{((\delta_1 + \delta_2) k^2_1 + \delta_1 \delta_3 k^2_2)}} k \ln (a) \coth_a (\xi) \right)^2 d\xi,
\]

(25)

where

\[
\xi = k_1 x + k_2 y - 2 (k_2 l_2 + k_0 k_1 l_1) t
\]

In particular, Setting
\[
A_1 = A_2 = k_1 = k_2 = l_1 = l_2 = p = q = k = \delta_0 = \delta_1 = \delta_2 = \delta_3 = B_2 = 1, B_1 = 0, a = e,
\]

We obtain new exact solution to (14).

\[
u(\xi) = 2 \sqrt{\frac{2}{3}} \coth (\xi)
\]

(26)

\[
W(\xi) = 2 \sqrt{\frac{2}{3}} \coth (\xi) . \exp (i \vartheta)
\]

(27)

\[
F(\xi) = \frac{1}{2} \int \left( 2 \sqrt{\frac{2}{3}} \coth (\xi) \right)^2 d\xi = \left( \frac{4}{3} \right) (\xi - \coth (\xi))
\]

(28)

where

\[
\xi = x + y - 4t, \quad \vartheta = x + y - 6t,
\]

See Figure (1) and (2): where \( \xi := x + y - 2. \quad t = 0.5 \)
Case 2:

\begin{align*}
\alpha_0 &= \alpha_0, A_2 = A_2, k_1 = k_1, k_2 = k_2, l_1 = l_1, l_2 = l_2, B_1 = B_1, B_2 = 0 \quad (29) \\
A_1 &= -\sqrt{\frac{((\delta_1 + \delta_2) k_1^2 + \delta_1 \delta_3 k_2^2)}{2 (k_1^2 + \delta_3 k_2^2) (\delta_0 k_1^2 + k_2^2)}} \left( \frac{\alpha_0 A_2}{k \ln(a)} \right) \\
\alpha_1 &= \sqrt{\frac{2 (k_1^2 + \delta_3 k_2^2) (\delta_0 k_1^2 + k_2^2)}{((\delta_1 + \delta_2) k_1^2 + \delta_1 \delta_3 k_2^2)}} \left( \frac{B_1}{A_2} k \ln(a) \right) \\
l_3 &= -2k^2 \ln^2(a) (\delta_0 k_1^2 + k_2^2) - \delta_0 l_1^2 - l_2^2
\end{align*}

Substituting (28) into (20), we have

\[ u(\xi) = \sqrt{\frac{2 (k_1^2 + \delta_3 k_2^2) (\delta_0 k_1^2 + k_2^2)}{((\delta_1 + \delta_2) k_1^2 + \delta_1 \delta_3 k_2^2)}} \frac{k \ln(a)}{\coth_a(\xi)} \cdot \exp \left\{ i \left[ l_1 x + l_2 y - \left( 2k^2 \ln^2(a) (\delta_0 k_1^2 + k_2^2) + \delta_0 l_1^2 + l_2^2 \right) t \right] \right\} \]

In view of (15), (18) and (29), we obtain the general solution of Eq.(14) in the form

\[ W = \sqrt{\frac{2 (k_1^2 + \delta_3 k_2^2) (\delta_0 k_1^2 + k_2^2)}{((\delta_1 + \delta_2) k_1^2 + \delta_1 \delta_3 k_2^2)}} \frac{k \ln(a)}{\coth_a(\xi)} \cdot \exp \left\{ i \left[ l_1 x + l_2 y - \left( 2k^2 \ln^2(a) (\delta_0 k_1^2 + k_2^2) + \delta_0 l_1^2 + l_2^2 \right) t \right] \right\} \]
and

\[
F(\xi) = v(\xi) = \int \frac{k_1}{(k_1^2 + \delta_2 k_2^2)} \left( \sqrt{\frac{2(k_2^2 + \delta_3 k_2^2)(\delta_2 k_1^2 + k_2^2)}{((\delta_1 + \delta_2) k_1^2 + \delta_1 \delta_2 k_2^2) \coth_a(\xi)}} \right)^2 d\xi, \tag{32}
\]

where

\[
\xi = k_1 x + k_2 y - 2(k_2 l_2 + \delta_0 k_1 l_1) t
\]

In particular, Setting

\[A_2 = B_1 = k_1 = k_2 = l_1 = l_2 = \alpha_0 = \delta_0 = \delta_1 = \delta_2 = \delta_3 = p = q = k = 1, B_2 = 0, a = e,\]

We obtain new exact solution to (14).

\[
u(\xi) = 2 \sqrt{\frac{2}{3}} \left( \frac{1}{\coth(\xi)} \right) \tag{33}
\]

\[
W(\xi) = 2 \sqrt{\frac{2}{3}} \left( \frac{1}{\coth(\xi)} \right) \cdot \exp(i\vartheta) \tag{34}
\]

\[
F(\xi) = \frac{1}{2} \int \left( 2 \sqrt{\frac{2}{3}} \left( \frac{1}{\coth(\xi)} \right) \right)^2 d\xi = \left( \frac{4}{3} \right) (\xi - \tanh(\xi)) \tag{35}
\]

where

\[
\xi = x + y - 4t, \quad \vartheta = x + y - 6t,
\]

See Figure (3) and (4): where \(\xi = x + y - 4, \; t = 1\).
Figure 3: $|W(\xi)| = \left| 2\sqrt{\frac{2}{3}} \left( \frac{1}{\coth(\xi)} \right) \right|$

Figure 4: $F(\xi) = \left( \frac{4}{3} \right) (\xi - \tanh(\xi))$

Case 3:

$$\alpha_0 = \alpha_0, \alpha_1 = \alpha_1, B_2 = B_2, B_1 = -B_2, l_1 = l_1, l_2 = l_2, k_1 = k_1, A_2 = A_2$$ (36)

$$A_1 = \left( \frac{2\alpha_0 B_2 + \alpha_1 A_2}{\alpha_1} \right), \quad k_2 = \sqrt{-\frac{\delta_1 + \delta_2}{\delta_1 \delta_3} k_1}$$

$$l_3 = (4\delta_0 k^2 k_1^2 \ln^2(a) - l_2^2 - \delta_0 l_1^2) - 4k^2 k_1^2 \ln^2(a) \left( \frac{\delta_1 + \delta_2}{\delta_1 \delta_3} \right)$$

Substituting (35) into (20), we have

$$u(\xi) = \left( \frac{\alpha_0 B_2 + \alpha_1 A_2}{B_2} \right) \left( \frac{1 + \coth_a(\xi)}{1 - \coth_a(\xi)} \right); \quad \xi \neq -\left( \frac{\ln \left( \frac{\xi}{q} \right)}{2k \ln(a)} \right); \quad q \neq 0.$$ (37)

In view of (15), (18) and (36), we obtain the general solution of Eq.(14) in the form

$$W = \left( \frac{\alpha_0 B_2 + \alpha_1 A_2}{B_2} \right) \left( \frac{1 + \coth_a(\xi)}{1 - \coth_a(\xi)} \right)$$

$$\times \exp \left\{ i \left[ l_1 x + l_2 y + \left( 4\delta_0 k^2 k_1^2 \ln^2(a) - l_2^2 - \delta_0 l_1^2 \right) - 4k^2 k_1^2 \ln^2(a) \left( \frac{\delta_1 + \delta_2}{\delta_1 \delta_3} \right) \right] t \right\}$$

and

$$F(\xi) = v(\xi) = \int \frac{k_1}{(k_1^2 + \delta_3 k_2^2)} \left( \left( \frac{\alpha_0 B_2 + \alpha_1 A_2}{B_2} \right) \left( \frac{1 + \coth_a(\xi)}{1 - \coth_a(\xi)} \right) \right)^2 d\xi.$$ (39)
where
\[ \xi = k_1 \left( x + \sqrt{-\frac{\delta_1 + \delta_2}{\delta_1 \delta_3}} y - 2 \left( \sqrt{-\frac{\delta_1 + \delta_2}{\delta_1 \delta_3}} l_2 + \delta_0 l_1 \right) t \right) \]

In particular, Setting

\[ \alpha_0 = \alpha_1 = B_2 = l_1 = l_2 = k_1 = p = q = \delta_3 = 1 \]
\[ A_2 = k = \delta_0 = \delta_1 = 1, \delta_2 = -2, B_1 = -1, a = e \]

We obtain new exact solution to (14).

\[ u(\xi) = -2 \left( \frac{1 + \coth(\xi)}{1 - \coth(\xi)} \right) \]  
(40)

\[ W(\xi) = -2 \left( \frac{1 + \coth(\xi)}{1 - \coth(\xi)} \right) \exp(i\vartheta) \]  
(41)

\[ F(\xi) = \frac{1}{2} \int \left( -2 \left( \frac{1 + \coth(\xi)}{1 - \coth(\xi)} \right) \right)^2 d\xi = \left( \frac{1}{2} \right) \left( \frac{\coth(\xi) + 1}{\coth(\xi) - 1} \right)^2 \]  
(42)

where
\[ \xi = x + y - 4t, \vartheta = x + y + 6t, \]

See Figure (5) and (6) : where \( \xi = x + y - 0.12; t = 0.03. \)
Case 4:

\[ \alpha_0 = \alpha_0, \alpha_1 = \alpha_1, B_2 = B_2, B_1 = B_1, l_1 = l_1, l_2 = l_2, k_1 = k_1, A_2 = A_2 \quad (43) \]

\[ A_1 = -\left( \frac{2\alpha_0 B_2 + \alpha_1 A_2}{\alpha_1} \right), \quad k_2 = \sqrt{-\frac{\delta_1 + \delta_2}{\delta_1 \delta_3} k_1} \]

\[ l_3 = (4\delta_0 k_2^2 k_1^2 \ln^2(a) - l_2^2 - \delta_0 l_1^2) - 4k_2^2 k_1^2 \ln^2(a) \left( \frac{\delta_1 + \delta_2}{\delta_1 \delta_3} \right) \]

Substituting (42) into (20), we have

\[ u(\xi) = \left( \frac{\alpha_0 B_2 + \alpha_1 A_2}{B_2} \right) \left( \frac{1 - \coth_a(\xi)}{1 + \coth_a(\xi)} \right) ; \xi \neq -\left( \frac{\ln \left( \frac{p}{q} \right)}{2k \ln(a)} \right) ; p \neq 0. \quad (44) \]

In view of (15), (18) and (43), we obtain the general solution of Eq. (14) in the form

\[ W = \left( \frac{\alpha_0 B_2 + \alpha_1 A_2}{B_2} \right) \left( \frac{1 - \coth_a(\xi)}{1 + \coth_a(\xi)} \right) \quad (45) \]

\[ \times \exp \left\{ i \left[ l_1 x + l_2 y + \left( (4\delta_0 k_2^2 k_1^2 \ln^2(a) - l_2^2 - \delta_0 l_1^2) - 4k_2^2 k_1^2 \ln^2(a) \left( \frac{\delta_1 + \delta_2}{\delta_1 \delta_3} \right) \right) t \right\} \]

and

\[ F(\xi) = v(\xi) = \int \frac{k_1}{(k_1^2 + \delta_3 k_2^2)} \left( \left( \frac{\alpha_0 B_2 + \alpha_1 A_2}{B_2} \right) \left( \frac{1 - \coth_a(\xi)}{1 + \coth_a(\xi)} \right) \right)^2 d\xi, \quad (46) \]

where

\[ \xi = k_1 \left( x + \sqrt{-\frac{\delta_1 + \delta_2}{\delta_1 \delta_3} y - 2 \left( \sqrt{-\frac{\delta_1 + \delta_2}{\delta_1 \delta_3} l_2 + \delta_0 l_1} \right) t \right) \]

In particular, Setting

\[ \alpha_0 = l_1 = l_2 = k_1 = \delta_1 = q = 1, \delta_2 = -2, \delta_3 = \frac{1}{4} \]

\[ k = \alpha_1 = B_2 = B_1 = p = 2, A_2 = 3, a = e^{-2}, \delta_0 = -1, \]

We obtain new exact solution to (14).

\[ u(\xi) = 4 \left( 1 - \left( \frac{2(e^{-2})^2 + (e^{-2})(e^{-2})}{2(e^{-2})^2 - (e^{-2})(e^{-2})} \right) \right) \]

\[ \left( 1 + \left( \frac{2(e^{-2})^2 + (e^{-2})(e^{-2})}{2(e^{-2})^2 - (e^{-2})(e^{-2})} \right) \right) \quad (47) \]
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\[
W(\xi) = \left[ 4 \left( 1 - \frac{2(e^{-2})(2\xi) + (e^{-2})(-2\xi)}{2(e^{-2})(2\xi) - (e^{-2})(-2\xi)} \right) \right] \cdot \exp(i\vartheta)
\]

(48)

\[
F(\xi) = \frac{1}{2} \int \left[ 4 \left( 1 - \frac{2(e^{-2})(2\xi) + (e^{-2})(-2\xi)}{2(e^{-2})(2\xi) - (e^{-2})(-2\xi)} \right) \right]^2 d\xi = \frac{1}{8} (e^{16\xi})
\]

(49)

where

\[
\xi = x + 2y - 2t, \ \vartheta = x + y + 192t,
\]

See Figure (7) and (8): where \( \xi = x - 2t + 4; \ y = 2. \)

\[
\text{Figure 7: } |W(\xi)| = \left| 4 \left( 1 - \frac{2(e^{-2})(2\xi) + (e^{-2})(-2\xi)}{2(e^{-2})(2\xi) - (e^{-2})(-2\xi)} \right) \right|
\]

\[
\text{Figure 8: } F(\xi) = \frac{1}{8} (e^{16\xi})
\]

5  Comparison between the \( \coth_a(\xi) \) method and some existing methods

In this section we will compare the solutions obtained by the \( \coth_a(\xi) \) method and the following three existing methods: \( \left( \frac{\vartheta}{\xi} \right) \) expansion method, the Jacobi Elliptic Functions
method and \textit{cosine} method. The comparison will be limited to the solution $u(\xi)$ of equation (19) since this equation has been obtained using a transformation from the main equation (14).

**Proposition 1** Some solutions obtained by solving equation (19) using \left(\frac{\xi}{\xi}\right) \text{ expansion method} can also be generated by the proposed \textit{coth}_a(\xi) method.

**Proof:** We solved the equation (19) using \left(\frac{\xi}{\xi}\right) expansion method, and obtained three types of solutions one of which is:

$$u(\xi) = \left(\sqrt{\frac{2(k_2^2 + k_3^2 \delta_0)(k_1^2 + \delta_3 k_3^2)}{\delta_1 \delta_3 k_2^2 + (\delta_1 + \delta_2) k_1^2}}\right) \left(\sqrt{\frac{\lambda^2 - 4\mu}{2}}\right) \left(\frac{c_1 \cosh \left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right) + c_2 \sinh \left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right)}{c_1 \sinh \left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right) + c_2 \cosh \left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right)}\right)$$  \hspace{1cm} \text{(50)}

where $(\lambda^2 - 4\mu) > 0; \xi = k_1 x + k_2 y - 2(k_3 l_2 + \delta_0 k_1 l_1) t$. The above solution is exactly the solution (22) in the case of $a = e; k = 1; p = 1; q = 1; c_2 = 0; \lambda^2 - 4\mu = 4$, and the solution (29) in the case of $a = e; k = 1; p = 1; q = 1; c_1 = 0; \lambda^2 - 4\mu = 4$. \hfill \Box

**Proposition 2** There are some solutions of equation (19), generated by our \textit{coth}_a(\xi) method, which cannot be obtained by the \left(\frac{\xi}{\xi}\right) expansion method.

**Proof:** Let us consider the following values of parameters of equations (22), (29), (36) and (43): $l_1 = l_2 = k_1 = k_2 = \delta_0 = \delta_1 = \delta_2 = \delta_3 = \alpha_0 = \alpha_1 = A_2 = B_2 = 1; k = \pi; a = 2; p = 5; q = 7$. Thus the following solutions cannot be obtained by \left(\frac{\xi}{\xi}\right) expansion method:

$$u(\xi) = \sqrt{\frac{8}{3}} \pi \ln(2) \cdot \left(\frac{5.2^{(\pi \xi)} + 7.2^{(-\pi \xi)}}{5.2^{(\pi \xi)} - 7.2^{(-\pi \xi)}}\right)$$

$$u(\xi) = \sqrt{\frac{8}{3}} \pi \ln(2) \cdot \left(\frac{5.2^{(\pi \xi)} - 7.2^{(-\pi \xi)}}{5.2^{(\pi \xi)} + 7.2^{(-\pi \xi)}}\right)$$

$$u(\xi) = \frac{1 + \left(\frac{5.2^{(\pi \xi)} + 7.2^{(-\pi \xi)}}{5.2^{(\pi \xi)} - 7.2^{(-\pi \xi)}}\right)}{1 - \left(\frac{5.2^{(\pi \xi)} + 7.2^{(-\pi \xi)}}{5.2^{(\pi \xi)} - 7.2^{(-\pi \xi)}}\right)}$$

$$u(\xi) = \frac{1 - \left(\frac{5.2^{(\pi \xi)} + 7.2^{(-\pi \xi)}}{5.2^{(\pi \xi)} - 7.2^{(-\pi \xi)}}\right)}{1 + \left(\frac{5.2^{(\pi \xi)} + 7.2^{(-\pi \xi)}}{5.2^{(\pi \xi)} - 7.2^{(-\pi \xi)}}\right)}$$
where $\xi = x + y - 4t$. 

**Proposition 3** Some solutions obtained by solving equation (19) using Jacobi Elliptic Functions method with the auxiliary differential equation $F^2(\xi) = P F^4(\xi) + Q F^2(\xi) + R$, can also be generated by the proposed $\text{coth}_a(\xi)$ method.

**Proof:** The Jacobi Elliptic Functions method generates many types of solutions to equation (19). One of these solutions is the following:

$$u(\xi) = \sqrt{2m^2} \sqrt{\frac{(k_2^2 + k_1^2 \delta_0)(k_1^2 + \delta_3 k_2^2)}{(\delta_1 + \delta_2) k_1^2 + \delta_1 \delta_3 k_2^2}} sn(\xi) \quad (51)$$

$$\lim_{m \to 1} u(\xi) = \sqrt{2} \sqrt{\frac{(k_2^2 + k_1^2 \delta_0)(k_1^2 + \delta_3 k_2^2)}{(\delta_1 + \delta_2) k_1^2 + \delta_1 \delta_3 k_2^2}} \tanh(\xi) \quad (52)$$

where $\xi = k_1 x + k_2 y - 2(k_2 l_2 + \delta_0 k_1 l_1) t$. This solution is exactly the solution (29) in the case of $a = e; k = 1; p = 1; q = 1, m \to 1$. A second solution of Jacobi Elliptic Functions method is:

$$u(\xi) = \sqrt{2} \sqrt{\frac{(k_2^2 + k_1^2 \delta_0)(k_1^2 + \delta_3 k_2^2)}{(\delta_1 + \delta_2) k_1^2 + \delta_1 \delta_3 k_2^2}} ns(\xi) \quad (53)$$

$$\lim_{m \to 1} u(\xi) = \sqrt{2} \sqrt{\frac{(k_2^2 + k_1^2 \delta_0)(k_1^2 + \delta_3 k_2^2)}{(\delta_1 + \delta_2) k_1^2 + \delta_1 \delta_3 k_2^2}} \coth(\xi) \quad (54)$$

where $\xi = k_1 x + k_2 y - 2(k_2 l_2 + \delta_0 k_1 l_1) t$. This solution is similar to the solution (22) in the case of $a = e; k = 1; p = 1; q = 1, m \to 1$. 

**Proposition 4** There are some solutions of equation (19), generated by our $\text{coth}_a(\xi)$ method, which cannot be obtained by the Jacobi Elliptic Functions method.

**Proof:** Let us consider the following values of parameters of equations (22), (29), (36) and (43): $l_1 = l_2 = k_1 = k_2 = \delta_0 = \delta_1 = \delta_2 = \delta_3 = \alpha_0 = \alpha_1 = A_2 = B_2 = 1; k = \pi e; a = \sin(1); p = 2; q = 3$. Thus the following solutions cannot be obtained by the Jacobi Elliptic Functions method:
\[ u(\xi) = \sqrt{\frac{8}{3}} \pi^{\epsilon} \ln(\sin(1)). \left( \frac{2 \sin(1)^{\pi \epsilon \xi} + 3 \sin(1)^{\pi \epsilon \xi} - 3 \sin(1)^{\pi \epsilon \xi}}{2 \sin(1)^{\pi \epsilon \xi} + 3 \sin(1)^{\pi \epsilon \xi}} \right) \]

where \( \xi = x + y - 4t \).

**Proposition 5** All solutions generated by our \( \coth_{\alpha}(\xi) \) method cannot be obtained by the cosine method.

**Proof:** The only solution of equation (19) obtained by the cosine method where:

\[
\begin{align*}
    u(x, y, t) &= \begin{cases} 
        \lambda \cos(\mu \xi); & |\xi| < \frac{\pi}{\mu} \\
        0; & \text{otherwise}
    \end{cases}
\end{align*}
\]

is:

\[ u(\xi) = \mu \sqrt{2} \frac{(k_{\xi_{1}}^{2} + k_{\xi_{1}}^{2} \delta_{0})(k_{\xi_{1}}^{2} + \delta_{3} k_{\xi_{2}}^{2})}{\delta_{1} + \delta_{2}} \frac{1}{\cos(\mu \xi)} \]

where \( \xi = k_{1}x + k_{2}y - 2(k_{2}l_{2} + \delta_{0}k_{1}l_{1}) t \)

It is obvious that this solution is different from all types of solutions generated by our \( \coth_{\alpha}(\xi) \) method. \( \square \)

### 6 Conclusion

In this article we have proposed a new method, called \( \coth_{\alpha}(\xi) \) Expansion method, to obtain the exact solutions for Davey-Stewartson equation. The comparison of the solutions
generated by our method with the ones obtained by some existing methods shows the efficiency of our method. The method can be applied to many other nonlinear partial differential equations in mathematical physics.

References


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